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ON A THEOREM OF KY FAN

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1. Introduction. Let H be a Hilbert space over the field of complex numbers C and let $\mathscr{L}(H)$ be the collection of all bounded linear operators in H.

Given a compact operator $T \in \mathscr{L}(H)$ we denote by $(s_n(T))$ the sequence of singular numbers of T, i.e., the eigenvalues of the positive compact operator $[T^* T]^{1/2}$, each one repeated a number of times equal to its multiplicity and ordered so that $s_1(T) \geqslant s_2(T) \geqslant \ldots \geqslant 0$. If $[T^* T]^{1/2}$ has less than n eigenvalues, then we put $s_n(T) = 0$.

The following inequality on the singular numbers of two compact operators $T, R \in \mathscr{L}(H)$ was proved by Ky Fan [1] in 1951:

$$(1) \qquad \sum_{j=1}^{n} s_j(T+R) \leqslant \sum_{j=1}^{n} s_j(T) + \sum_{j=1}^{n} s_j(R), \qquad n = 1, 2, \ldots$$

Since the singular numbers of a compact operator T acting between two Hilbert spaces H and K coincide with its approximation numbers

$$(2) \qquad a_n(T) = \inf\{\|T - T_n\|: \ T_n \in \mathscr{L}(H, K),\ \operatorname{rank}(T_n) < n\}$$

(see [3], Chapter 11), and the right part of (2) makes sense for every operator T acting between Banach spaces, it is natural to wonder if the inequality of Fan also holds in the Banach space setting.

This question has a negative answer as one can see in the books by Pietsch [3] or [4]. Just like it happens with the inequality of Weyl [5] between the eigenvalues and the singular numbers of a compact operator (see [2]), it turns out that inequality (1) holds in Banach spaces with an additional constant.

The aim of this note is to prove this fact by using very elementary techniques.

2. The inequality of Ky Fan. We start with an example which shows that in the Banach space setting at least an additional constant is necessary in Fan's inequality.

Let l_∞^2 be the space C^2 with the max-norm and let

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}: l_\infty^2 \to l_\infty^2$$

and

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}: l_\infty^2 \to l_\infty^2.$$

Clearly, $a_1(T) = 4$ and $a_1(R) = 2$. Moreover, as the operators

$$T_1 = \begin{pmatrix} -1/4 & -1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 1/3 & 1 \\ 2/3 & 2 \end{pmatrix}$$

are of rank one, we have

$$a_2(T) \leqslant \|T - T_1\| = 5/4 \quad \text{and} \quad a_2(R) \leqslant \|R - R_1\| = 2/3.$$

But

$$T+R = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix},$$

whence $a_1(T+R) = 6$ and $a_2(T+R) = 2$. Consequently, we obtain

$$\sum_{j=1}^{2} a_j(T+R) \nleqslant \sum_{j=1}^{2} a_j(T) + \sum_{j=1}^{2} a_j(R).$$

Let us show now that (1) holds in Banach spaces with the additional constant 2.

THEOREM. *Let E and F be Banach spaces and let $T, R \in \mathscr{L}(E, F)$. Then*

$$\sum_{j=1}^{n} a_j(T+R) \leqslant 2\Big(\sum_{j=1}^{n} a_j(T) + \sum_{j=1}^{n} a_j(R)\Big), \qquad n = 1, 2, \ldots$$

Proof. Given n, choose k such that $n \leqslant 2k-1 \leqslant n+1$. By the additivity of the approximation numbers, we have

$$\sum_{j=1}^{n} a_j(T+R) \leqslant 2\sum_{j=1}^{k} a_{2j-1}(T+R) \leqslant 2\Big(\sum_{j=1}^{k} a_j(T) + \sum_{j=1}^{k} a_j(R)\Big)$$
$$\leqslant 2\Big(\sum_{j=1}^{n} a_j(T) + \sum_{j=1}^{n} a_j(R)\Big).$$

Finally, let us remark that the constant 2 is the best possible (see [4], Proposition 2.3.7).

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