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WEAKLY COMPACT OPERATORS FROM A B-SPACE INTO THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

BY

CHARLES SWARTZ (TUCSON, ARIZONA)

In [6], Diestel has obtained a representation theorem for any bounded linear operator from a B-space X into the space $L^p(\Sigma, \mu: Y)$ of Bochner p-summable Y-valued functions which generalizes Theorem VI, 8.1, of [7] for $p = 1$. In [7] there is also given a characterization of the weakly compact operators from X into $L^1(\Sigma, \mu)$. In this note we characterize the weakly compact operators from X into $L^1(\Sigma, \mu: Y)$ for certain B-spaces Y (Theorem 2) and compare our result with the scalar case as given in Theorem VI, 8.1, of [7].

Our notation and terminology will be that of [7] unless otherwise indicated. Throughout the paper X and Y will denote B-spaces and (S, Σ) a measurable space. If μ is a positive finite measure on (S, Σ), $L^1(\Sigma, \mu: X)$ denotes the space of all strongly measurable X-valued functions $f: S \to X$ such that

$$\|f\|_1 = \int_S \|f(t)\| d\mu(t) < \infty$$

equipped with the norm $\| \ \|_1$ (see [7], III. 3). Our first result qives a characterization of relatively weak compact subsets of $L^1(\Sigma, \mu: X)$ for certain B-spaces X. This theorem generalizes Theorem IV, 8.9, of [7], and the proof is quite similar to that given in [7] for the scalar case.

If X is a B-space and μ is a finite positive measure on (S, Σ), X is said *to have the Radon-Nikodym property with respect to* (S, Σ, μ) if whenever $m: \Sigma \to X$ is a vector measure (see [7], IV. 10) of bounded variation which is absolutely continuous with respect to μ, there is a function $f \in L^1(\Sigma, \mu: X)$ such that (see [4])

$$m(E) = \int_E f d\mu \quad \text{for each } E \in \Sigma.$$

For use below we note that X has the Radon-Nikodym property with respect to (S, Σ, μ) iff X has the Radon-Nikodym property with respect to (S, Σ_0, μ) for each separable sub-σ-algebra Σ_0 of Σ (see [4], the corollary of Theorem 1 and the discussion in Section 3, p. 26). There is a detailed discussion of the Radon-Nikodym property in Section 3 of [4], and a list of B-spaces which have the Radon-Nikodym property with respect to any measure space in the corollary of Theorem 7 of [4].

We now give a characterization of the relatively weakly compact subsets of $L^1(\Sigma, \mu: X)$ when X has the Radon-Nikodym property with respect to (S, Σ, μ).

THEOREM 1. *Let μ be a finite measure on (S, Σ) and suppose that X and X' have the Radon-Nikodym property with respect to (S, Σ, μ). The following three conditions are equivalent:*

I. *$K \subseteq L^1(\Sigma, \mu: X)$ is relatively weakly compact;*

II. (i) *K is bounded;*

(ii) *the family of vector measures $\{\int_{\cdot} f d\mu : f \in K\}$ is uniformly countably additive;*

(iii) *for each $A \in \Sigma$, $\{\int_A f d\mu : f \in K\}$ is relatively weakly compact in X;*

III. *K satisfies* (i) *and* (iii) *of* II *and*

(ii)′ *the family of scalar measures $\{\int_{\cdot} \|f\| d\mu : f \in K\}$ is uniformly countably additive.*

Proof. Suppose K is relatively weakly compact. We establish III. Condition (i) is clear. If (ii)′ fails to hold, there is an $\varepsilon > 0$, a sequence $\{E_n\}$ of pairwise disjoint sets from Σ, a sequence $\{f_m\} \subseteq K$, sequences of positive integers $\{i_n\}$ and $\{j_n\}$ with $i_n < j_n < i_{n+1}$ and

$$\sum_{k=i_n}^{j_n} \int_{E_k} \|f_n\| d\mu > \varepsilon \quad \text{for all } n. \tag{1}$$

For each n, set

$$H_n = \bigcup_{k=i_n}^{j_n} E_k.$$

Then (1) becomes

$$\int_{H_n} \|f_n\| d\mu > \varepsilon \quad \text{for all } n. \tag{2}$$

From (2), for each n, there is a $g_n: S \to X'$ such that g_n is μ-essentially bounded on H_n by 1, $g_n(t) = 0$ for $t \in S \setminus H_n$ and

$$\int_{H_n} \langle f_n(t), g_n(t) \rangle d\mu(t) > \varepsilon. \tag{3}$$

(Recall the dual of $L^1\big(H_n, \Sigma(H_n), \mu: X\big)$ is $L^\infty\big(H_n, \Sigma(H_n), \mu: X'\big)$ when X' has the Radon-Nikodym property.) Set

$$g = \sum_{n=1}^{\infty} g_n$$

and note that since the $\{H_n\}$ are pairwise disjoint, g is μ-essentially bounded on S by 1. Thus, the linear map $f \to \langle f, g \rangle$ from $L^1(\Sigma, \mu: X)$ into $L^1(\Sigma, \mu)$ is continuous with respect to the norm topologies of both spaces and, hence, continuous with respect to the weak topologies of both spaces (see [7], V. 3.15). Therefore, $\{\langle f_n, g \rangle\}$ is relatively weakly compact in $L^1(\Sigma, \mu)$, and $\{|\langle f_n, g \rangle|\}$ is also relatively weakly compact in $L^1(\Sigma, \mu)$ (see [7], IV. 8.10). By Theorem IV. 8.9, of [7], the countable additivity of the integrals $\int_E |\langle f_n, g \rangle| d\mu$ is uniform with respect to n. But this contradicts (3), and (ii)′ must hold.

For (iii), note that, for each $A \in \Sigma$, the map

$$f \to \int_A f d\mu$$

from $L^1(\Sigma, \mu: X)$ into X is linear and continuous with respect to the norm topologies of both spaces and, therefore, continuous with respect to the weak topologies of the spaces. Hence, if K is relatively weakly compact, the set in (iii) must also be relatively weakly compact being the image of K under the above-mentioned map. Thus we have shown that I implies III.

It is clear that III implies II. So it suffices to show II implies I. Let $\{f_n\}$ be a sequence in K. By [7], III. 8.5 (and the proof of this lemma), there is a sub-σ-field Σ_1 of Σ which is generated by a countable field Σ_0 such that $f_n \in L^1(S, \Sigma_1, \mu_1: X)$, where μ_1 is μ restricted to Σ_1. By (iii), for each $E \in \Sigma_0$, the sequence $\{\int_E f_n d\mu\}$ has a weakly convergent subsequence. Since Σ_0 is countable, by a diagonalization procedure, we can find a subsequence of $\{f_n\}$ (which we continue to denote by $\{f_n\}$) such that the sequence $\{\int_E f_n d\mu\}$ converges weakly in X for each $E \in \Sigma_0$. By [7], IV. 8.8, and (ii), for each $x' \in X'$ and $E \in \Sigma_1$, $\lim \langle x', \int_E f_n d\mu \rangle$ exists. Therefore, using (iii), we infer that, for each $E \in \Sigma_1$, the sequence $\{\int_E f_n d\mu\}$ converges weakly in X. If we set

$$\lambda(E) = \lim \int_E f_n d\mu \qquad \text{for each } E \in \Sigma_1$$

(limit in the weak topology), then $x'\lambda$ is a measure for each $x' \in X'$ (see [7], III. 7.4) and, therefore, λ is a vector measure (see [7], IV. 10.1).

We also claim that λ has bounded variation. For if $\{E_i : i = 1, \dots, n\}$ is a partition of S by subsets of Σ_1, we have

$$\begin{aligned}\sum_{i=1}^{n} \|\lambda(E_i)\| &= \sum_{i=1}^{n} \sup\left\{\left|\lim_m \left\langle x', \int_{E_i} f_m d\mu \right\rangle\right| : \|x'\| \leqslant 1\right\} \\ &\leqslant \sum_{i=1}^{n} \underline{\lim}_{m} \int_{E_i} \|f_m\| d\mu \leqslant \underline{\lim}_{m} \sum_{i=1}^{n} \int_{E_i} \|f_m\| d\mu \leqslant B,\end{aligned}$$

where B is the bound for K given by (i).

Now $\mu(E) = 0$, $E \in \Sigma_1$, implies $\lambda(E) = 0$, so by the Radon-Nikodym property there is an $f \in L^1(\Sigma_1, \mu_1 : X)$ such that

$$\lambda(E) = \int_E f d\mu \quad \text{for } E \in \Sigma_1.$$

Thus, we infer that, for any simple function $g \in L^\infty(\Sigma_1, \mu_1 : X')$,

$$\lim_n \int_S \langle f_n, g \rangle d\mu = \int_S \langle f, g \rangle d\mu.$$

Since the simple functions are dense in $L^\infty(\Sigma_1, \mu_1 : X')$, by (i) and [1], IX. 1.1, the sequence $\{f_n\}$ converges weakly to f in $L^1(\Sigma_1, \mu_1 : X)$. Since $L^1(\Sigma_1, \mu_1 : X)$ is a linear subspace of $L^1(\Sigma, \mu : X)$, $\{f_n\}$ actually converges to f in $L^1(\Sigma, \mu : X)$. By the Šmul'yan-Eberlein Theorem (see [7], V. 6.1), K is relatively weakly compact.

Remark. The equivalence of I and II is the analogue of IV. 8.9, of [7], and the equivalence of I and III can be compared to IV. 8.10, of [7].

It should be noted that the paper [5] contains criteria for weak compactness somewhat different from those in Theorem 1 for the case where X is reflexive.

Let $T: X \to L^1(\Sigma, \mu : Y)$ be linear and continuous. Then there is a finitely additive set function $m : \Sigma \to L(X, Y)$ (here $L(X, Y)$ denotes the space of bounded linear operators from X into Y) such that for each $x \in X$ the vector measure $m(\cdot)x$ has bounded variation with

$$v\big(m(\cdot)x\big) = \|Tx\|_1$$

(here $v\big(m(\cdot)x\big)$ denotes the variation of $m(\cdot)x$) and $Tx = dm(\cdot)x/d\mu$ (see [6]). Now T is weakly compact iff $\{Tx : \|x\| \leqslant 1\}$ is relatively weakly compact in $L^1(\Sigma, \mu : Y)$, so, by Theorem 1, we have

THEOREM 2. *Let Y and Y' have the Radon-Nikodym property with respect to (S, Σ, μ). Then T is weakly compact iff*

(a) *the family of scalar measures $\{v\big(m(\cdot)x\big) : \|x\| \leqslant 1\}$ is uniformly countably additive;*

(b) *for each $A \in \Sigma$, $m(A) \in L(X, Y)$ is weakly compact.*

Proof. Condition (a) is condition (ii)′ since

$$v\big(m(A)x\big) = \int_A \|Tx\| d\mu,$$

and condition (b) is just condition (iii) since

$$m(A)x = \int_A Tx d\mu.$$

We would like to compare this result with the criteria for weak compactness of a map U from X into $L^1(\Sigma, \mu)$, the space of scalar-valued

μ-integrable functions. Recall in this case U has a representation of the form

$$Ux = \frac{da(\cdot)x}{d\mu},$$

where $a\colon \Sigma \to X'$, and U is weakly compact iff a is countably additive with respect to the norm topology of X' (see [7], VI. 8.1). Condition (b) is trivially satisfied in this case since $Y = \boldsymbol{R}$ so that one might conjecture that condition (a) is equivalent to $m\colon \Sigma \to L(X, Y)$ being countably additive with respect to the norm topology of $L(X, Y)$. We show that this is not in general true, but rather that condition (a) is equivalent to a type of bounded-multiplier convergence as discussed in [9].

First we observe that condition (a) implies m is countably additive with respect to the norm topology of $L(X, Y)$.

PROPOSITION 3. *Let $m\colon \Sigma \to L(X, Y)$ be finitely additive and such that, for $x \in X$, $m(\cdot)x$ is a vector measure of bounded variation. If condition* (a) *of Theorem 2 is satisfied, then m is countably additive with respect to the norm topology.*

Proof. Let $\{E_i\}$ be a pairwise disjoint sequence from Σ. Then, for any N, M $(M \geqslant N)$, we have

$$\Big\|\sum_{j=N}^{M} m(E_j)\Big\| = \sup_{\|x\| \leqslant 1} \Big\|\sum_{j=N}^{M} m(E_j)x\Big\| \leqslant \sup_{\|x\| \leqslant 1} \sum_{j=N}^{M} v\big(m(E_j)x\big),$$

and the term on the right goes to zero uniformly for $\|x\| \leqslant 1$ as $N, M \to \infty$, by condition (a).

PROPOSITION 4. *Let $m\colon \Sigma \to L(X, Y)$ be finitely additive and such that, for each $x \in X$, $m(\cdot)x$ is a vector measure of bounded variation. The following conditions are equivalent:*

(i) *the family of scalar measures $\{v(m(\cdot)x)\colon \|x\| \leqslant 1\}$ is uniformly countably additive;*

(ii) *for each pairwise disjoint sequence $\{E_j\} \subseteq \Sigma$, the series $\sum_j \|m(E_j)x\|$ converges uniformly for $\|x\| \leqslant 1$;*

(iii) *for each pairwise disjoint sequence $\{E_i\} \subseteq \Sigma$, the series $\sum_j m(E_j)' y_j'$ converges uniformly in X' for $\|y_j'\| \leqslant 1$, $y_j' \in Y'$;*

(iv) *for each pairwise disjoint sequence $\{E_j\} \subseteq \Sigma$ and $\{y_j'\} \subseteq Y'$, $\|y_j'\| \leqslant 1$, the series $\sum_j m(E_j)' y_j'$ converges unconditionally in X'.*

Proof. The inequality $\|m(E_j)x\| \leqslant v\big(m(E_j)x\big)$ shows that (i) implies (ii). To see that (ii) implies (iii) note that, for $M > N$,

$$\Big\|\sum_{i=N}^{M} m(E_j)' y_j'\Big\| = \sup_{\|x\| \leqslant 1} \Big|\sum_{j=N}^{M} \langle m(E_j)' y_j', x\rangle\Big| \leqslant \sup_{\|x\| < 1} \sum_{j=N}^{M} \|m(E_j)x\|.$$

Clearly, (iii) implies (iv). We next show (iv) implies (ii). If (ii) fails to hold, there are sequences $\{E_j\} \subseteq \Sigma$ (pairwise disjoint), $\{x_j\} \subseteq X$ with $\|x_j\| \leqslant 1$, $\{N_i\}$ and $\{M_i\}$ positive integers with $N_i < M_i < N_{i+1}$, and $\varepsilon > 0$ such that

$$\sum_{j=N_i}^{M_i} \|m(E_j)x_i\| > \varepsilon \quad \text{for all } i.$$

Pick $y'_j \in Y'$, $\|y'_j\| \leqslant 1$, such that

$$\sum_{j=N_i}^{M_i} |\langle y'_j, m(E_j)x_i\rangle| > \varepsilon \quad \text{for all } i. \tag{4}$$

But, by (iv),

$$\lim_{N\to\infty} \sum_{j=N}^{\infty} |\langle m(E_j)' y'_j, x\rangle| = 0$$

uniformly for $\|x\| \leqslant 1$ (see [8], condition (H)) which contradicts (4).

Finally, we show (ii) implies (i). If (i) fails to hold, there exist a pairwise disjoint sequence $\{E_j\} \subseteq \Sigma$, $\varepsilon > 0$, $\{x_j\} \subseteq X$ with $\|x_j\| \leqslant 1$, $\{N_i\}$ and $\{M_i\}$ sequences of positive integers with $N_i < M_i < N_{i+1}$ such that

$$\sum_{j=N_i}^{M_i} v\big(m(E_j)x_i\big) > \varepsilon \quad \text{for all } i.$$

For each j there is a partition $\{B_{j,k}: k = 1, \ldots, k_j\}$ of E_j such that

$$\sum_{j=N_i}^{M_i} \sum_{k=1}^{k_j} \|m(B_{j,k})x_i\| > \varepsilon \quad \text{for all } i. \tag{5}$$

But, by (ii), the series

$$\sum_{i=1}^{\infty} \sum_{j=N_i}^{M_i} \sum_{k=1}^{k_j} \|m(B_{j,k})x\|$$

converges uniformly for $\|x\| \leqslant 1$ which contradicts (5).

Remark. The results are quite similar to those in Theorem 6 of [2]; they are of a dual nature to the results of Batt. Conditions (iii) and (iv) are a type of bounded multiplier convergence as discussed in [9].

We now give an example which shows that the conditions of Proposition 4 are not equivalent to the countable additivity of the vector measure. For this example recall that a series $\sum x_n$ in a B-space X is *weakly unconditionally convergent* (w.u.c.) iff $\sum |\langle x', x_n\rangle| < \infty$ for each $x' \in X'$ [3], and the series $\sum x_n$ is *unconditionally convergent* (u.c.) iff every rearrangement converges in the norm topology of X (see [3] and [8]). (These terms often have different meanings in other papers.)

Example. Pick a B-space X and a series $\sum x_n$ in X such that $\sum x_n$ is w.u.c., not u.c., and $\|x_n\| \to 0$. For each positive integer n, define T_n: $X' \to c_0$ by $T_n x' = \langle x', x_n \rangle e_n$, where $e_n \in c_0$ is the vector with 1 in the n-th coordinate and 0 in the other coordinates.

Let S denote the positive integers and let Σ be the σ-algebra of all subsets of S. Define $m: \Sigma \to L(X', c_0)$ by

$$m(E) = \sum_{n \in E} T_n.$$

Since

$$\begin{aligned} \|m(E)\| &= \sup\{|\langle x', x_n \rangle| : n \in E, \|x'\| \leqslant 1\} \\ &= \sup\{\|x_n\| : n \in E\} \end{aligned}$$

and $\|x_n\| \to 0$, m is countably additive in the norm topology of $L(X', c_0)$. For each $x \in X'$, $E \in \Sigma$,

$$v\big(m(E)x'\big) = \sum_{n \in E} |\langle x', x_n \rangle|, \tag{6}$$

so that each $m(\cdot)x'$ has finite variation since $\sum x_n$ is w.u.c. But from (6) and the fact that $\sum x_n$ is not u.c., it follows that the family of scalar measures $\{v\big(m(\cdot)x'\big): \|x'\| \leqslant 1\}$ is not uniformly countably additive (see [8], condition (H)). That is, m does not satisfy condition (i) of Proposition 4.

To see that a series satisfying the above-mentioned conditions actually exists, consider $X = C[0, 1]$ with the usual sup norm. Define $x_n \in X$ by

$$x_n(t) = t^2/(1+t^2)^n \quad \text{for } 0 \leqslant t \leqslant 1,\ n \geqslant 0.$$

The series $\sum_{n=0}^{\infty} x_n$ converges pointwise to the function

$$f(t) = \begin{cases} 0, & t = 0, \\ 1+t^2, & 0 < t \leqslant 1. \end{cases}$$

For each $t \in [0, 1]$ and $N \geqslant 0$,

$$\Big|\sum_{n=0}^{N} x_n(t)\Big| \leqslant 2$$

so that $\sum x_n$ is w.u.c. Since the limit function f is not continuous, $\sum x_n$ is not u.c. Also $\|x_n\| \to 0$, so the series $\sum x_n$ satisfies the above-mentioned conditions.

In conclusion we remark that it would be desirable to lift the Radon-Nikodym assumption from Theorem 1. However, even in the scalar case (see [7], IV. 8) the Radon-Nikodym Theorem seems to be an indispensable tool.

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