COLLOQUIUM MATHEMATICUM

VOL. XL 1978 FASC. 1

ON σ-CONNECTED SPACES

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1. Introduction. A *continuum* is a compact connected metric space. A continuum X is called *semi-aposyndetic* provided for each two points of X there exists a subcontinuum of X which contains one of them in its interior and which does not contain the other one. A continuum X is *locally connected* provided every point of X has a neighbourhood base consisting of connected subsets of X. Clearly, each locally connected continuum is semi-aposyndetic.

A topological space X is said to be *σ-connected* (*weakly σ-connected*) provided it is connected and cannot be decomposed into countably infinitely many mutually separated (mutually separated, connected) non-empty subsets. A topological space X is said to be a *semi-continuum* provided each two points of X can be joined by means of a continuum contained in X. By Sierpiński's theorem, each continuum is a σ-connected space. It can easily be proved that each semi-continuum is a σ-connected space (see [1], p. 216).

The purpose of this note* is to answer in the affirmative the following two questions the first of which is due to Grispolakis et al. ([1], P 976) and the second one to Mycielski [4], Problem 3.

QUESTION 1. Suppose that X is a continuum such that each σ-connected subset of X is a semi-continuum. Is every connected subset of X arcwise connected?

QUESTION 2. Does every infinite σ-connected set contain an infinite proper σ-connected subset?

2. σ-connectedness. The following theorems concerning σ-connectedness are analogous to some theorems on connectedness in [2] (Theorem 4, p. 133, Theorem 5, p. 140, Theorem 7, p. 141, and Theorem 7, p. 249).

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* This work is a part of the author's doctoral dissertation and was done under the direction of Professor E. D. Tymchatyn.

THEOREM 1. *If C is a σ-connected subset of a σ-connected space X and*

$$X \setminus C = \bigcup_{i=1}^{\infty} M_i,$$

where the sets M_i $(i = 1, 2, 3, \ldots)$ are pairwise separated, then the set $C \cup \bigcup_{i=2}^{\infty} M_i$ is σ-connected.

Proof. We can assume that C is non-empty. Just suppose that

$$C \cup \bigcup_{i=2}^{\infty} M_i = A_1 \cup A_2 \cup A_3 \cup \ldots,$$

where the sets A_i $(i = 1, 2, 3, \ldots)$ are pairwise separated. Since the set C is σ-connected, we may assume that $C \subseteq A_1$. Thus

$$\bigcup_{i=2}^{\infty} A_i \subseteq \bigcup_{i=2}^{\infty} M_i.$$

Now,

$$X = M_1 \cup A_1 \cup \bigcup_{i=2}^{\infty} A_i = (M_1 \cup A_1) \cup \bigcup_{i,j=2}^{\infty} (A_i \cap M_j).$$

Since the sets A_i $(i = 2, 3, 4, \ldots)$ are pairwise separated, and the sets M_j $(j = 2, 3, 4, \ldots)$ are pairwise separated, the sets $A_i \cap M_j$ are pairwise separated. Also, the set $M_1 \cup A_1$ is separated from each of the sets $A_i \cap M_j$ $(i \geqslant 2$ and $j \geqslant 2)$. Indeed,

$$\begin{aligned}
\overline{(M_1 \cup A_1)} \cap (A_i \cap M_j) &= (\overline{M}_1 \cup \overline{A}_1) \cap (A_i \cap M_j) \\
&= (\overline{M}_1 \cap A_i \cap M_j) \cup (\overline{A}_1 \cap A_i \cap M_j) = \emptyset, \\
(M_1 \cup A_1) \cap \overline{(A_i \cap M_j)} &\subseteq (M_1 \cup A_1) \cap (\overline{A}_i \cap \overline{M}_j) \\
&= (M_1 \cap \overline{A}_i \cap \overline{M}_j) \cup (A_1 \cap \overline{A}_i \cap \overline{M}_j) = \emptyset.
\end{aligned}$$

Thus, the union

$$(M_1 \cup A_1) \cup \bigcup_{i,j=2}^{\infty} (A_i \cap M_j)$$

is a decomposition of X into a countable number of mutually separated sets. Since X is σ-connected and $A_1 \cup M_1 \neq \emptyset$, we have $A_2 = A_3 = A_4 = \ldots \ldots = \emptyset$. The connectedness of this set follows immediately from the above argument if the sets $A_3, A_4, \ldots$ are assumed to be empty. This completes the proof that $C \cup \bigcup_{i=2}^{\infty} M_i$ is σ-connected.

COROLLARY 1. *Every σ-connected set X which contains more than two points can be written as the union of two proper non-degenerate σ-connected subsets. If X is metric, then these sets may be taken to be F_σ.*

Proof. We have to consider two cases.

Case 1. There exists an element $x \in X$ such that $X \setminus \{x\}$ is not σ-connected. Hence

$$X \setminus \{x\} = A_1 \cup A_2 \cup A_3 \ldots,$$

where the sets A_i are mutually separated for each $i = 1, 2, 3, \ldots$, and at least two of the sets A_i are non-empty. Suppose that A_1 and A_2 are non-empty. Let

$$B_1 = \{x\} \cup \bigcup_{i=2}^{\infty} A_i \quad \text{and} \quad B_2 = \{x\} \cup \bigcup_{\substack{i=1 \\ i \neq 2}}^{\infty} A_i.$$

By Theorem 1, B_1 and B_2 are σ-connected and $X = B_1 \cup B_2$.

Case 2. For each $x \in X$, $X \setminus \{x\}$ is σ-connected. Let $x_1 \neq x_2$ be two elements of X. Then

$$X = (X \setminus \{x_1\}) \cup (X \setminus \{x_2\}).$$

The other assertions of the theorem are clear.

Mycielski has proved in [4] that there exists an infinite connected set which does not contain an infinite σ-connected subset.

COROLLARY 2. *Every infinite σ-connected set contains an infinite proper σ-connected subset.*

COROLLARY 3. *Let A and B be two closed (or two open) sets. If $A \cup B$ and $A \cap B$ are σ-connected, then A and B are also σ-connected.*

Proof. In case where A and B are closed (or open), the sets $A \setminus B$ and $B \setminus A$ are mutually separated. By Theorem 1, the sets

$$A = (A \setminus B) \cup (A \cap B) \quad \text{and} \quad B = (B \setminus A) \cup (A \cap B)$$

are σ-connected.

THEOREM 2. *Let X be a σ-connected space. If A is a σ-connected subset of X and C is a σ-component of $X \setminus A$, then $X \setminus C$ is σ-connected.*

Proof. Just suppose that

$$X \setminus C = \bigcup_{i=1}^{\infty} M_i,$$

where the sets M_i are pairwise separated for each $i = 1, 2, 3, \ldots$ Since the set A is σ-connected, and $A \subseteq X \setminus C$, we may assume that $A \subseteq M_1$. Thus

$$A \cap \left(C \cup \bigcup_{i=2}^{\infty} M_i\right) = \emptyset.$$

Hence

$$C \subseteq C \cup \bigcup_{i=2}^{\infty} M_i \subseteq X \setminus A.$$

Since C is a σ-component of $X\setminus A$, and $C\cup\bigcup_{i=2}^{\infty} M_i$ is σ-connected by Theorem 1,

$$C = C\cup\bigcup_{i=2}^{\infty} M_i.$$

Hence $M_2 = M_3 = M_4 = \ldots = \emptyset$. This completes the proof that $X\setminus C$ is σ-connected.

THEOREM 3. *In a σ-connected space X, let S be an infinite family of disjoint σ-connected subsets of X. If S_0 and S_1 are two arbitrary elements of S, then in $X\setminus S_0$ or in $X\setminus S_1$ there exists a σ-connected set which contains infinitely many elements of S. Moreover, this set may be taken to be F_σ provided X is metric and each of the sets S_0 and S_1 is a closed subset of X.*

Proof. Let C_j $(j = 0, 1)$ be the σ-component of $X\setminus S_j$ which contains S_{1-j}. Thus, $S_j \subseteq C_{1-j} \subseteq X\setminus S_{1-j}$. This implies that

$$S_{1-j} \subseteq X\setminus C_{1-j} \subseteq X\setminus S_j.$$

By Theorem 2, $X\setminus C_{1-j}$ is σ-connected. Since C_j is the σ-component of $X\setminus S_j$ which contains S_{1-j}, we have $X\setminus C_{1-j} \subseteq C_j$. If C_0 contains only a finite number of elements of S, then there exist infinitely many elements of S contained in $X\setminus C_0$ and, therefore, in C_1. This completes the proof that at least one of the sets C_0 and C_1 must contain infinitely many elements of the family S. If X is metric and each of the sets S_0 and S_1 is closed in X, then, clearly, C_j $(j = 0, 1)$ is an F_σ-set, since it is a closed subset of an open set.

3. Continua.

Definition. A continuum is said to be *finitely Suslinian* provided, for each number $\varepsilon > 0$, every collection of mutually disjoint subcontinua of X with diameters greater than ε is finite.

We denote the boundary and the interior of a set E by $\mathrm{bd}(E)$ and $\mathrm{Int}(E)$, respectively.

THEOREM 4. *If the continuum X is not locally connected, it contains a σ-connected F_σ-subset which is not a semi-continuum.*

Proof. Suppose that X is not locally connected at some point $p \in X$. Then there exists a closed neighbourhood E of p such that p is not contained in the interior of the component C of E containing p. Thus $p \in \overline{E\setminus C}$. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in $E\setminus C$ converging to p. For each $n = 1, 2, 3, \ldots$, let C_n be the component of E containing p_n. It follows that $C_n\cap C = \emptyset$, and $C_n\cap\mathrm{bd}(E) \neq \emptyset$. Without loss of generality, we may assume that the sequence $\{C_n\}_{n=1}^{\infty}$ consists of different components of E and is convergent to a subcontinnum L contained in C. It follows that $L\cap\mathrm{bd}(E) \neq \emptyset$.

Let F be a closed neighbourhood of p which is contained in the interior of E and let I be the component of F containing p. Since L is connected, we have

$$L\setminus[F\cup\mathrm{bd}(E)]\neq\emptyset.$$

Let $q\in\big(L\cap\mathrm{Int}(E)\big)\setminus F$. Let H be a closed neighbourhood of q such that $H\subseteq E\setminus F$. Let D be the component of H containing q. It follows that $D\cup I\subseteq C$ and $D\cap I=\emptyset$.

Therefore, the sets $D, I, C_1, C_2, \ldots$ form a family of disjoint σ-connected closed subsets of the σ-connected space X. By Theorem 3, there exists a σ-connected F_σ-subset W which contains infinitely many of the members of the family, and such that either $W\subseteq X\setminus D$ or $W\subseteq X\setminus I$. By symmetry, we may assume that $W\subseteq X\setminus I$. Clearly, p is a limit point of W, and hence the set $P=W\cup\{p\}$ is σ-connected F_σ.

It remains to show that P is not a semi-continuum. Just suppose that P is a semi-continuum. Let the point y be an element of $P\setminus F$. Since P is a semi-continuum, there exists a continuum $R\subseteq P$ joining p and y. Let T be the component of $R\cap F$ which contains p. Since R is a continuum, and F is a neighbourhood of p, we have $T\cap\mathrm{bd}(F)\neq\emptyset$. Hence $T\neq\{p\}$. But $p\in T\subseteq F$ implies $T\subseteq I$, and hence $T\cap W=\emptyset$. Thus $T=\{p\}$. This contradiction completes the proof of the theorem.

COROLLARY 4. *If X is a continuum such that each σ-connected subset of X is a semi-continuum, then the connected sets in X are arcwise connected.*

Proof. As shown in [1], it suffices to prove that the continuum X which fulfils the condition of Corollary 2 is semi-aposyndetic. But, by Theorem 4, X is locally connected, and hence it is semi-aposyndetic.

The following theorem was proved in [1] (Theorem 3.2) with the additional hypothesis that X is semi-aposyndetic.

THEOREM 5. *A continuum X is finitely Suslinian if and only if X satisfies (any) one of the following conditions:*

(i) *Every connected F_σ-subset of X is arcwise connected.*

(ii) *Every connected F_σ-subset of X is a semi-continuum.*

(iii) *Every σ-connected F_σ-subset of X is arcwise connected.*

(iv) *Every σ-connected F_σ-subset of X is a semi-continuum.*

(v) *Every weakly σ-connected F_σ-subset of X is arcwise connected.*

(vi) *Every weakly σ-connected F_σ-subset of X is a semi-continuum.*

Proof. The hypothesis that X is semi-aposyndetic is only used to show that (iv) implies that X is finitely Suslinian. This follows directly from Theorem 4 and from the fact that each locally connected continuum is semi-aposyndetic.

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Reçu par la Rédaction le 4. 6. 1976;
en version modifiée le 15. 4. 1977