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A DOMINATION THEOREM FOR FUNCTION ALGEBRAS

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1. Introduction. Let A be a commutative complex Banach algebra with unit element e. In [1] Arens has formulated a theorem stating that if for given elements $u, v \in A$ we have $\|ux\| \leqslant \|vx\|$ for all elements $x \in A$, then there exist an extension $B \supset A$ and an element $b \in B$ such that $u = bv$.

There arises an important problem (cf. [2]) whether a similar statement is true for several elements of A, i.e. whether the relation

$$\|ux\| \leqslant \|v_1x\| + \ldots + \|v_nx\| \tag{1}$$

holding for all $x \in A$, where $u, v_1, \ldots, v_n$ are fixed elements of A, implies that in some extension $B \supset A$ there are elements $b_1, b_2, \ldots, b_n \in B$ such that

$$u = b_1v_1 + \ldots + b_nv_n \tag{2}$$

(see Added in proof (i)).

If relation (1) holds true, possibly with a positive constant factor on the right-hand side, we say that the element u is *dominated by the elements* $v_1, v_2, \ldots, v_n$ (see Added in proof (ii)).

In this paper we show that the answer to this problem is in the affirmative in the case where A is a function algebra, and this is the contents of the domination theorem proved in Section 3.

2. Prerequisites. Let A and B be commutative complex unital Banach algebras. We say that B is an *extension* of A if there exists a unital isometric isomorphism of A into B. We shall treat such an isomorphism as an imbedding and write simply $A \subset B$.

Let $\mathfrak{M}(A)$ be the maximal ideal space of A. We say that A is a *function algebra* (a *uniform* or *sup-norm algebra*) if the norm of A is given by

$$\|x\| = \max_{M \in \mathfrak{M}(A)} |x(M)|,$$

where $x(M)$ is the Gelfand transform of an element $x \in A$. The *Šilov* *boundary* $\Gamma(A)$ of A is the smallest closed subset of $\mathfrak{M}(A)$ with the property

$$\|x\| = \max_{M\in\Gamma(A)} |x(M)| \quad \text{for all } x \in A.$$

In the sequel we shall need the following characterization of the Šilov boundary:

An ideal $M_0 \in \mathfrak{M}(A)$ is in $\Gamma(A)$ if and only if for each $\varepsilon > 0$ and each neighbourhood U of M_0 in $\mathfrak{M}(A)$ there exists an element $x_\varepsilon \in A$ such that

$$\|x_\varepsilon\| = 1 \quad \text{and} \quad |x_\varepsilon(M)| < \varepsilon \text{ for } M \notin U. \tag{3}$$

For the details the reader is referred to [3].

3. The domination theorem. Our main result reads as follows:

THEOREM. *Let A be a unital function algebra and let $u, v_1, \ldots, v_n \in A$. Then there exist an extension $B \supset A$ and elements $b_1, b_2, \ldots, b_n \in B$ such that relation* (2) *holds true if and only if u is dominated by $v_1, v_2, \ldots, v_n$, i.e. there exists a constant $C > 0$ such that*

$$\|ux\| \leqslant C(\|v_1x\| + \ldots + \|v_nx\|) \quad \textit{for all } x \in A. \tag{4}$$

Proof. If for some extension $B \supset A$ relation (2) holds true, then for each $x \in A$

$$\|ux\| = \|b_1v_1x + \ldots + b_nv_nx\| \leqslant \|b_1\|\,\|v_1x\| + \ldots + \|b_n\|\,\|v_nx\|,$$

and relation (4) is satisfied with $C = \max\{\|b_2\|, \ldots, \|b_n\|\}$.

Conversely, suppose that relation (4) holds true. We shall show that this implies the existence of a positive constant a such that for every maximal ideal M in the Šilov boundary $\Gamma(A)$ we have

$$|u(M)| \leqslant a\big(|v_1(M)| + \ldots + |v_n(M)|\big). \tag{5}$$

In fact, suppose that (5) fails while (4) holds true. In this case, for each positive integer k there is an ideal $M_k \in \Gamma(A)$ such that

$$|u(M_k)| > k\big(|v_1(M_k)| + \ldots + |v_n(M_k)|\big) \quad \text{for } k = 1, 2, \ldots \tag{6}$$

We fix an integer k satisfying

$$k > 2C, \tag{7}$$

where C is the constant in relation (4). For any ε satisfying $0 < \varepsilon < 1$, by continuity of the functions $u(M), v_1(M), \ldots, v_n(M)$ there is a neighbourhood U_ε of M_k in $\mathfrak{M}(A)$ such that

$$|u(M)| > k\big(|v_1(M)| + \ldots + |v_n(M)|\big) \tag{8}$$

and

$$|v_i(M_k)| + \varepsilon > |v_i(M)| \geqslant \frac{1}{2}\,|v_i(M_k)|, \quad i = 1, 2, \ldots, n, \tag{9}$$

for all $M \in U_\varepsilon$. For the same ε we find by (3) an element $x_\varepsilon \in A$ with $\|x_\varepsilon\| = 1$ and $|x_\varepsilon(M)| < \varepsilon$ for $M \notin U_\varepsilon$. Thus $|x_\varepsilon(M_0)| = 1$ for some $M_0 \in \Gamma(A)$ and, since $\varepsilon < 1$, we have $M_0 \in U_\varepsilon$. Using this fact and relations (9), (8) and (4) we obtain

$$\frac{k}{2}\sum_{i=1}^{n} |v_i(M_k)| \leqslant k \sum_{i=1}^{n} |v_i(M_0)| < |u(M_0)|$$

$$= |u(M_0)x_\varepsilon(M_0)| \leqslant \|ux_\varepsilon\| \leqslant C \sum_{i=1}^{n} \|v_i x_\varepsilon\|$$

$$= C \sum_{i=1}^{n} \max\{\sup_{M\in U_\varepsilon} |v_i(M)x_\varepsilon(M)|, \sup_{M\in\Gamma(A)\setminus U_\varepsilon} |v_i(M)x_\varepsilon(M)|\}$$

$$\leqslant C \sum_{i=1}^{n} \max\{|v_i(M_k)| + \varepsilon, \varepsilon\|v_i\|\} \leqslant C \sum_{i=1}^{n} |v_i(M_k)| + C\varepsilon\Big(n + \sum_{i=1}^{n} \|v_i\|\Big)$$

or

$$\sum_{i=1}^{n} |v_i(M_k)| \leqslant \frac{2C}{k} \sum_{i=1}^{n} |v_i(M_k)| + \frac{2C}{k}\varepsilon\Big(n + \sum_{i=1}^{n} \|v_i\|\Big).$$

Since this holds for an arbitrary ε, we conclude that

$$\sum_{i=1}^{n} |v_i(M_k)| \leqslant \frac{2C}{k} \sum_{i=1}^{n} |v_i(M_k)|,$$

which, in view of $k > 2C$, implies

$$\sum_{i=1}^{n} |v_i(M_k)| = 0.$$

We shall show now that $u(M_k) = 0$ which, in view of (6), will give the desired contradiction establishing relation (5). To this end suppose that $|u(M_k)| = \delta > 0$. Choose an ε satisfying

$$0 < \varepsilon < \min\Big\{1, \delta\Big[2C\Big(n + \sum_{i=1}^{n} \|v_i\|\Big)\Big]^{-1}\Big\} \tag{10}$$

and take a neighbourhood U of M_k such that

$$|v_i(M)| < \varepsilon \tag{11}$$

and

$$\frac{\delta}{2} < |u(M)| \tag{12}$$

for all $M \in U$.

As before, we find an element $x \in A$ satisfying $\|x\| = 1$ and $|x(M)| < \varepsilon$ for each $M \notin U$. So $|x(M_0)| = 1$ for some $M_0 \in U$, and thus, as before, taking into account (11) and (12), we have

$$\frac{\delta}{2} < |u(M_0)| = |u(M_0)x(M_0)| \leqslant \|ux\| \leqslant C\sum_{i=1}^{n} \|v_i x\|$$

$$\leqslant C\sum_{i=1}^{n} (\varepsilon + \varepsilon\|v_i\|) = \varepsilon C\Big(n + \sum_{i=1}^{n} \|v_i\|\Big).$$

This gives a contradiction with relation (10), and so we have established formula (5).

Formula (5) implies now

$$|u(M)| = \beta(M)\sum_{i=1}^{n} |v_i(M)| \quad \text{for } M \in \Gamma(A), \tag{13}$$

where $0 \leqslant \beta(M) \leqslant a$.

Write

$$u(M) = |u(M)|\exp[i\varphi(M)]$$

and

$$v_i(M) = |v_i(M)|\exp[i\varphi_i(M)] \quad \text{for } i = 1, 2, \ldots, n.$$

Thus (13) implies

$$u(M) = \sum_{i=1}^{n} b_i(M)v_i(M) \quad \text{for } M \in \Gamma(A), \tag{14}$$

where

$$b_i(M) = \beta(M)\exp\big[i\big(\varphi(M) - \varphi_i(M)\big)\big], \tag{15}$$

so that

$$\sup_{M\in\Gamma(A)} |b_i(M)| = \sup_{\Gamma(A)} \beta(M) \leqslant a.$$

Define B as the algebra of all bounded complex-valued functions on $\Gamma(A)$. This is clearly a Banach algebra and an extension of A; moreover, the functions b_i given by (15) are in B and, as shown by (14), satisfy relation (2). Thus our result is established

Remark. Generally speaking, the functions b_i occurring in relation (2) are not continuous on $\Gamma(A)$ in the topology inherited from $\mathfrak{M}(A)$. A simple example is obtained by setting $A = C[0, 1]$, $u(t) = t\sin t^{-1}$ and $v(t) = t$.

Added in proof. (i) A negative answer to the problem on p. 317 for general Banach algebras has been given in the paper of V. Müller, *On domination and extensions of Banach algebras*, Studia Mathematica 73 (in print).

(ii) More on domination is given in the paper of W. Żelazko, *On domination and separation of ideals in commutative Banach algebras*, Studia Mathematica 71 (1981) (in print).

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