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BOREL STRUCTURES FOR FUNCTION SPACES

BY

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1. Introduction. Let $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ be two Borel spaces and F a collection of measurable functions from X to Y. There is then a natural map $\Phi\colon F\times X \to Y$ defined by $\Phi(f, x) = f(x)$. Let us say that F is admissible if there is a *Borel structure* $\boldsymbol{F}$ on F such that the map Φ from the product space $F\times X$ into Y is measurable; where, of course, the product space is equipped with the product σ-algebra $\boldsymbol{F}\times\boldsymbol{B}$. In that case we refer to $\boldsymbol{F}$ as an admissible structure for F. The purpose of this paper is to characterize admissible sets and to discuss as to how nice an admissible structure be chosen. This problem, in case both the Borel spaces are countably generated, was first considered by Aumann [1], p. 614. Theorem 1 of this paper is in essence proved by him — but by using complicated arguments. Our arguments, we hope, are elegant. However we rely heavily on the classical methods (see [2], [4], p. 207, and [5], p. 133). In Section 2 we give the notation and terminology. In Section 3 we consider the separable case. Finally, in Section 4, we consider the general case.

2. Notation and preliminaries. Assume throughout this section that $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ are separable Borel spaces (i.e., countably generated and containing singletons). Let $\boldsymbol{G} = [G_n, n\geqslant 1]$ be a countable generator for $\boldsymbol{B}$. Define the *Marczewski function* [4] f on X by

$$f(x) = \sum \frac{2\chi_{G_i}(x)}{3^i}.$$

Then f is an isomorphism between X and the range of f, where the latter is equipped with the relativized Borel algebra of I, the unit interval (see also [5]). Consequently, we can and shall take X to be a subset of I whenever necessary and $\boldsymbol{B}$ its relativized Borel algebra. Similar considerations hold for $(Y, \boldsymbol{C})$. Without explicit mention X, Y will be given the relative topologies, so that they become separable metric spaces.

Recall ([2], pp. 345, 373) that if X is a separable metric space, then open (closed) subsets of X are said to be *Borel sets of additive* (*multiplicative* respectively) *class* 0. Having defined classes upto a; $0 < a < \omega_1$, then define Borel sets of additive (multiplicative) class a to be countable unions of sets of the previous multiplicative (additive. resp.) classes. (Here ω_1 denotes the first uncountable ordinal). A function f from X to Y is said to be *of class* a if inverse image of every open subset of Y is a set of additive class a in X. We shall denote for $0 \leqslant a < \omega_1$, the set of Borel functions of class $\leqslant a$ by $C_a(X, Y)$. If no confusion can arise, we sometimes write C_a for $C_a(X, Y)$. Let us say that a collection F of functions from X to Y is of *bounded Borel class* if $F \subset C_a$ for some $a < \omega_1$. One might wonder that in identifying an arbitrary separable Borel space with a subset of I, via the Marczewski function, we have fixed a generator and that, consequently, the definition of bounded Borel class depends on the generators. But actually it is not so. If under some identification a collection F of maps from X to Y is of bounded Borel class, then it remains so under any other identification. This is a direct consequence of the composition laws for Borel classes (see [2], p. 376). It is however true that $C_a(X, Y)$ does depend on the generator chosen.

If $X = Y = I$, $a < \omega_1$, then we can find a Borel function $U_a(x, y)$ on $I \times I$ such that,

$$C_a(I, I) \subset \{U_a(x, \cdot);\, x \in I\}.$$

This is a direct consequence of the corresponding fact for Baire classification ([6], p. 137) and the connection between Borel and Baire classifications ([2], p. 393).

Axiom of choice has been assumed throughout this paper.

3. Countably generated case. Till further notice we assume that $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ are separable and F, a collection of measurable maps from X to Y.

THEOREM 1. *The following are equivalent*:

(i) *F is of bounded Borel class.*

(ii) *F is admissible.*

(iii) *There is a separable admissible structure for F.*

(iv) *The power set of F is an admissible structure for F.*

Proof of (i) $\Rightarrow$ (ii). Since obviously subsets of admissible sets are admissible, enough to show that $C_a(X, Y)$ is admissible for each $a < \omega_1$.

In case $X = Y = I$, let us choose a function U_a on $I \times I$ as mentioned in Section 2. Choose a subset Z of I such that the map

$$T : x \to U_a(x, \cdot)$$

is one to one on Z onto $C_a(I, I)$. Having thus identified $C_a(I, I)$ with Z via T, the relativized σ-algebra on Z can be brought to $C_a(I, I)$ in an obvious way. The measurability of U_a on $I\times I$ and hence on $Z\times I$ shows that the structure given to $C_a(I, I)$ is admissible.

In case $X\subset I$, $Y=I$, observe that any element of $C_a(X, I)$ can be regarded as the restriction to X of an element of $C_{a+1}(I, I)$ (see [2], p. 434 and 435). Thus $C_a(X, I)$ can be identified as a subset of $C_{a+1}(I, I)$. Since this latter set is admissible by the above para, so is every subset of it and hence $C_a(X, I)$ is admissible.

In case $X\subset I$, $Y\subset I$, observe that $C_a(X, Y)$ is a subset of $C_a(X, I)$ and since the latter set is admissible, so is the former.

In view of the remarks — via Marczewski function — made in Section 2, the proof is complete.

Proof of (ii) $\Rightarrow$ (i). First we show that if $(\Omega_0, \boldsymbol{B}_0)$ is a separable metric space with its Borel σ-algebra and $(\Omega_1, \boldsymbol{B}_1)$ is any measurable space and h is a measurable map from $\Omega_1\times\Omega_0$ to Ω_2 — another separable metric space, then there is an $a<\omega_1$ such that for each $x\in\Omega_1$, the map $h(x, \cdot)\colon \Omega_0\to\Omega_2$ is of class $\leqslant a$. Then our result follows by taking $\Omega_0=X$, $\Omega_1=F$ with any admissible structure, $\Omega_2=Y$ and $h=\varphi$.

Since Ω_2 is separable, by standard arguments one can find a countably generated $\boldsymbol{B}_1'\subset\boldsymbol{B}_1$ such that h is measurable w.r.t. $(\Omega_1\times\Omega_0, \boldsymbol{B}_1'\times\boldsymbol{B}_0)$ and thus there is no loss to assume that $\boldsymbol{B}_1$ is countably generated. Again if x, x' are in the same atom of $\boldsymbol{B}_1$ the functions $h(x, \cdot)$ and $h(x', \cdot)$ are the same and, consequently, there is no loss to assume that $\boldsymbol{B}_1$ is separable. In this case, by using again the Marczewski function associated with any countable generator for $\boldsymbol{B}_1$, one can think of $(\Omega_1, \boldsymbol{B}_1)$ as a separable metric space with its Borel algebra. Thus, h being a Borel map, it should be of some class, say a. Then, by [2], p. 377, every section of h is a function of class $\leqslant a$, as desired.

Thus (i) and (ii) are equivalent. Observe that (iii) implies (ii) and that from the arguments of (i) $\Rightarrow$ (ii) it is clear that (i) $\Rightarrow$ (iii). Since any structure larger than an admissible structure is admissible (iii) $\Rightarrow$ (iv) and (iv) implies (ii). This completely proves the theorem.

Two comments are in order at this point. First, observe that the above proof makes use of the axiom of choice. Aumann also makes use of it, though he does not explicitly state it (see especially the discussion following Lemma 4.1 of [1]). Secondly, the notion of bounded Borel class is the same as that of bounded Banach classes as discussed by Aumann.

One can now ask whether consistent structures can be given to $C_a(X, Y)$ for $0\leqslant a<\omega_1$. That is, a structure $\boldsymbol{C}_a$ to C_a such that

(i) $\boldsymbol{C}_a$ is separable,

(ii) $0\leqslant\beta<a$ implies $C_\beta\in\boldsymbol{C}_a$,

(iii) $0\leqslant\beta<a$ implies $\boldsymbol{C}_a|C_\beta=\boldsymbol{C}_\beta$.

The answer in the affirmative is given by Theorem 2. Let $C_\infty=\bigcup\limits_{a<\omega_1} C_a$.

THEOREM 2. *There is a separable structure* $\boldsymbol{C}_\infty$ *on* C_∞ *satisfying the following conditions*:

(i) $F \subset C_\infty$ *is admissible iff* $\boldsymbol{C}_\infty | F$ *is an admissible structure for* F.

(ii) *The structures* $\boldsymbol{C}_a = \boldsymbol{C}_\infty | C_a$ *for* $0 \leqslant a < \omega_1$ *are consistent in the above sense.*

Proof. Put $C_a^* = C_a - \bigcup_{\beta < a} C_\beta$.

Since, by theorem 1, C_a^* is admissible, fix separable structures $\boldsymbol{C}_a^*$ on C_a^*. By any Marczewski function identify $(C_a^*, \boldsymbol{C}_a^*)$ with a subspace of $\xi(a) \times I$, where ξ is any one to one map on the ordinals $< \omega_1$ into I. Thus C_a^* s being disjoint we have identified C_∞ as a subset of $I \times I$. Now the relativized Borel algebra on this subset serves the purpose [1].

Aumann defines an admissible structure $\boldsymbol{F}$ on an admissible set F to be natural if any other admissible structure on F contains $\boldsymbol{F}$. He also gives an example where such a natural structure need not exist. Observe that if one takes $C_0(I, I)$, the space of continuous functions on I to I, then it has a natural structure. In fact, its topological field (when C_0 is equipped with supremum metric) is the same as the field induced by the evaluation maps. In passing we note that if $F_1 \subset F_2$ and F_2 has a natural structure, say $\boldsymbol{F}_2$, then F_1 also has a natural structure, namely $\boldsymbol{F}_2 \cap F_1$. This remark leads us to believe that many subsets even in the case of the unit interval have no natural structure in the sense of Aumann. This leads us to define naturality in a different way. A separable admissible structure $\boldsymbol{F}$ on an admissible family F is *Blackwellian* if no proper separable substructure of $\boldsymbol{F}$ is admissible. It is clear that if the set Z, that occurs in the proof of Theorem 1, is a Blackwell space, then the structure we get on C_a in that theorem is Blackwellian (for details regarding Blackwell spaces, see [3]). Thus the existence of Blackwellian structures seems to be connected with Blackwellian selections. But, however, we feel that the existence or non-existence of such structures should better be treated directly rather than through selection theorems — as one knows that, in general, nice selections are difficult to obtain.

We conclude this section with a slight generalization of Theorem 1. Let now $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ be countably generated and F a collection of measurable maps from X to Y. By treating X and Y as pseudometric spaces or by looking at the canonical separable spaces (that is, the spaces of atoms with quotient structures), one can define the notion of bounded Borel classes and by applying Theorem 1 one can prove

THEOREM 3. *The following are equivalent*:

(i) F *is of bounded Borel class.*

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(1) The author's original proof of a part of this theorem used the continuum hypothesis. The present modification, without using it, is due to C. Ryll-Nardzewski and is represented here with his kind permission.

(ii) *F is admissible.*

(iii) *F has a countably generated admissible structure.*

(iv) *For F, its power set is an admissible structure.*

4. The general case. Let us assume $(X, \boldsymbol{B})$ to be any Borel space and $(Y, \boldsymbol{C})$ to be separable and F a collection of measurable maps from X to Y. Say that F is of *bounded Borel class* if there is a countably generated sub-algebra $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that

(i) f in F implies f is $\boldsymbol{B}_0$ - measurable;

(ii) F is of bounded Borel class w.r.t. $(X, \boldsymbol{B}_0)$ and $(Y, \boldsymbol{C})$.

We shall now prove the following

THEOREM 4. *The following are equivalent:*

(i) *F is of bounded Borel class.*

(ii) *F is admissible.*

(iii) *F has a countably generated admissible structure.*

(iv) *Power set of F is an admissible structure for $\boldsymbol{F}$.*

Proof. To show that (i) $\Rightarrow$ (ii), take a countably generated $\boldsymbol{B}_0 \subset \boldsymbol{B}$ as given by the boundedness of F and apply theorem 3 for the collection F from $(X, \boldsymbol{B}_0)$ to $(Y, \boldsymbol{C})$. Since $(Y, \boldsymbol{C})$ is separable (ii) $\Rightarrow$ (iii) is obvious (just make use of the fact that every set in a product σ-algebra is in the σ-algebra generated by a countable number of rectangles.) By using the same arguments, given a countably generated admissible structure $\boldsymbol{F}$ on F we can find a countably generated $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that the map φ is $\boldsymbol{F} \times \boldsymbol{B}_0$ measurable. This $\boldsymbol{B}_0$ is the required one to show that (iii) $\Rightarrow$ (i). Of course, (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) are trivial. This proves the theorem.

A theorem similar to the above can be stated in case $(Y, \boldsymbol{C})$ is countably generated and not necessarily separable — just as theorem 3 was formulated from theorem 1.

We now pass to the general case and do not assume $(Y, \boldsymbol{C})$ to be countably generated. Let $\boldsymbol{C}_0$ be any countably generated substructure of $\boldsymbol{C}$. We say that F is of *bounded Borel class relative to* $\boldsymbol{C}_0$ if the collection F regarded as maps from $(X, \boldsymbol{B})$ to $(Y, \boldsymbol{C}_0)$ is of bounded Borel class in the previous sense. We now have a complete solution to our problem in the following theorem:

THEOREM 5. *Let $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ be any two Borel spaces and F a collection of measurable maps from X to Y. Let $\boldsymbol{C}$ have a generator of cardinality $\leqslant \aleph$, an infinite cardinal. The following are then equivalent:*

(i) *F is bounded for any countably generated substructure of $\boldsymbol{C}$.*

(ii) *F is bounded for any finitely generated substructure of $\boldsymbol{C}$.*

(iii) *For F its power set is an admissible structure.*

(iv) *F is admissible.*

(v) *There is an $\aleph$ generated admissible structure for F.*

Proof. By using theorem 4 and standard arguments the following string of implications can be observed: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

Though the naturality or Blackwellian nature of structures can be defined even in the non-separable case, the unhappy state of affairs existing in the separable case has refrained us from doing so — for we had just to be content with the formulation and without even an epsilon insight into the problem.

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