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POLYNOMIAL STRUCTURES ON PRINCIPAL FIBRE BUNDLES

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Introduction. Yano [9] introduced the notion of an *f-structure* which is a $(1, 1)$-tensor field of constant rank on a C^∞-manifold M and satisfies the equality $f^3+f = 0$. This notion is a generalization of almost complex and almost contact structures and has been studied by several authors (see [3], [4] and [7]) with a particular focus on framed structures. In its turn, it has been generalized by Goldberg and Yano [2] who defined a polynomial structure of degree d which is a $(1, 1)$-tensor field of constant rank on M and satisfies the algebraic equation

$$Q(f) = f^d + a_d f^{d-1} + \ldots + a_2 f + a_1 I = 0,$$

where I is the identity mapping, and $f^{d-1}(x), \ldots, f(x)$, I are linearly independent for any $x \in M$. The polynomial Q is called *structural*, and f is called a *Q-structure*.

In this paper we consider a principal fibre bundle P over a C^∞ (paracompact) manifold M of dimension m, with a structural group G of dimension n and a projection $\pi\colon P\to M$, equipped with a connection Γ. In Sections 1 and 2 we show that $(1, 1)$-tensor fields on M and G may be lifted to tensor fields on P. In Section 3 we construct a Q-structure on P, which is obtained by lifting Q-structures on M and G. In Sections 4 and 5 we study integrability and normality of lifted Q-structures.

Similar problems for almost complex and f-structures have been considered by Ishihara and Yano [6], and Tanno [8].

In the sequel we denote by X, Y (with indices or not) arbitrary vectors of TP or vector fields on P, by Z, Z' etc. vector fields on M, and by A, B etc. left invariant vector fields on G. By Z^h we denote the horizontal lift of Z; thus Z^h is the horizontal vector field on P satisfying $\pi_* Z^h = Z$. A fundamental vector field on P with respect to A will be denoted by A^*. If ω is the form of a connection Γ, then $\omega(A^*) = A$ and $\pi_* A = 0$. The Lie algebra of G is denoted by $\mathfrak{g}$.

If V is an arbitrary C^∞-manifold and h is a $(1,1)$-tensor field on V, then we put

(1) $$[h,h](D,E) = [hD,hE]-h[D,hE]-h[hD,E]+h^2[D,E]$$

for every vector fields D, E on V. It is easy to verify that $[h,h]$ is a tensor field of type $(1,2)$ on V.

1. Horizontal lifts of $(1,1)$**-tensor fields.** Let f be a $(1,1)$-tensor field on M. Put

$$f^h(X) = (f\pi_* X)_p^h \quad \text{for } X \in P_p.$$

Then f^h is a $(1,1)$-tensor field on P, which will be called a *horizontal lift* of f. Of course, the rank of f^h at a point p of P is equal to the rank of f at $x = \pi(p)$. Therefore, if f is a tensor field of a constant rank, then the rank of f^h is constant.

LEMMA 1.1. (a) $f^h(X) = 0$ *for any vertical vector* X *and*

(2) $$f^h(Z^h) = (fZ)^h,$$

(b) f^h *is invariant by* G: $f^h \circ R_{g*} = R_{g*} \circ f^h$, *where* $g \in G$ *and* R_{g*} *is the differential of the mapping* $P \ni p \mapsto pg$.

Proof. Part (a) is obvious. If $X \in P_p$, then

$$f^h(R_{g*}X) = (f\pi_* R_{g*}X)_{pg}^h = (f\pi_* X)_{pg}^h = R_{g*}(f\pi_* X)_p^h = R_{g*}(f^h X),$$

since the connection Γ is invariant by G.

Since the formula

(3) $$F_\pi(Z) = \pi_*(FZ^h)$$

defines a $(1,1)$-tensor field on M and $(F_\pi)^h = F$, each G-invariant $(1,1)$-tensor field F on P, which vanishes on the vertical subbundle of TP and maps horizontal vectors into horizontal vectors, is a horizontal lift of some tensor field on M.

Remark. Formula (3) defines a tensor field F_π only if F is invariant by G and, in general, the equality $(F_\pi)^h = F$ does not hold. More precisely, $(F_\pi)^h(X) = hF(hX)$, and so that equality holds if and only if $F(hX) = 0$ and $hF(X) = F(hX)$. Of course, always $(f^h)_\pi = f$.

THEOREM 1.1. *If a connection* Γ *is flat, then*

(4) $$[f^h, f^h](Z^h, Z'^h) = \big([f,f](Z,Z')\big)^h.$$

Conversely, if $\operatorname{rank} f = m$ *and formula* (4) *holds for every* Z, Z', *then* Γ *is a flat connection.*

Proof. If Γ is flat, then $[Z^h, Z'^h] = [Z, Z']^h$ for any Z, Z'. Hence, by (1) and (2),

$$\begin{aligned}
&[f^h, f^h](Z^h, Z'^h) \\
&\quad = [(fZ)^h, (fZ')^h] - f^h[Z^h, (fZ')^h] - f^h[(fZ)^h, Z'^h] + (f^h)^2[Z^h, Z'^h] \\
&\quad = [fZ, fZ']^h - f^h([Z, fZ']^h) - f^h([fZ, Z']^h) + (f^h)^2([Z, Z']^h) \\
&\quad = ([f, f](Z, Z'))^h.
\end{aligned}$$

Conversely, if $\operatorname{rank} f = m$, then for any vector fields Z, Z' on M there are vector fields Z_1, Z_1' such that $Z = fZ_1$ and $Z' = fZ_1'$. It follows from (1) and (4) that

$$[Z^h, Z'^h] = ([f, f](Z_1, Z_1'))^h + f^h[Z^h, Z_1'^h] + f^h[Z_1^h, Z'^h] - (f^h)^2[Z_1^h, Z_1'^h].$$

Hence, the vector field $[Z^h, Z'^h]$ is horizontal and $[Z^h, Z'^h] = [Z, Z']^h$. Therefore Γ is a flat connection.

2. Fundamental tensor fields. A $(1, 1)$-tensor field F on P is said to be *fundamental* if F sends fundamental vector fields on P into fundamental vector fields and vanishes on the horizontal subbundle of TP.

If f is a left-invariant $(1, 1)$-tensor field on G, then $f(A)$ is a *left-invariant vector field* for every $A \in \mathfrak{g}$. Hence the formula

$$f^*(X) = (f\omega X)_p^*, \qquad X \in P_p,$$

defines a tensor field of type $(1, 1)$ on P. Clearly, $\operatorname{rank} f^* = \operatorname{rank} f$.

Proposition 2.1. *The correspondence $f \mapsto f^*$ is a bijection from the set of left-invariant $(1, 1)$-tensor fields on G to the set of fundamental tensor fields on P.*

Proof. Obviously,

$$(5) \qquad f^*(A^*) = (fA)^* \quad \text{for } A \in \mathfrak{g}$$

and $f(Z^h) = 0$. Thus f^* is fundamental. If F is fundamental, then the formula $f(A) = B$, where $F(A^*) = B^*$, defines a left-invariant tensor field on G such that $f^* = F$. If $f^* = f_1^*$, then $f(A)^* = f_1(A)^*$ and, consequently, $f(A) = f_1(A)$ for every A of $\mathfrak{g}$.

Proposition 2.2. *For any A, B of $\mathfrak{g}$ we have*

$$(6) \qquad [f^*, f^*](A^*, B^*) = ([f, f](A, B))^*.$$

Proof. The correspondence $A \mapsto A^*$ is a homomorphism of Lie algebras. Now formula (6) follows immediately from (1) and (5).

3. Polynomial structures. Goldberg and Petridis [1] have proved that a simply connected manifold M of dimension m is parallelizable if and only if there exists a polynomial structure on M with structure polynomial of degree m having m distinct non-zero real roots. We will prove here the following generalization of this result:

THEOREM 3.1. *If a manifold M is parallelizable, Q is a polynomial of degree $d \leqslant m = \dim M$ and m is even or Q has a real root, then there is a Q-structure on M.*

Proof. Let

$$Q(x) = x^k (x-a_1)^{l_1} \ldots (x-a_r)^{l_r} (x^2+b_1 x+c_1)^{m_1} \ldots (x^2+b_s x+c_s)^{m_s},$$

where $x-a_1, \ldots, x-a_r, x^2+b_1x+c_1, \ldots, x^2+b_sx+c_s$ are distinct irreducible polynomials over $\boldsymbol{R}$. Then

$$d = k+l_1+\ldots+l_r+2m_1+\ldots+2m_s \leqslant m.$$

Put

$$n_h = k+l_1+\ldots+l_{h-1} \qquad \text{for } h = 1, \ldots, r+1,$$

$$p_h = n_{r+1}+2m_1+\ldots+m_{h-1} \qquad \text{for } h = 1, \ldots, s+1,$$

$$\varepsilon_j = 2\left[\frac{j-1}{2}\right], \quad \delta_j = j+(-1)^{j-1} \qquad \text{for } j = 1, 2, \ldots,$$

$$e_h = -\frac{1}{2} b_h, \quad d_h = \frac{1}{4}(4c_h - b_h^2) \qquad \text{for } h = 1, \ldots, s.$$

Taking an arbitrary basis $Z_1, \ldots, Z_m$ of vector fields on M we also put

$$f_0(Z_i) = 0, \quad f_0(Z_j) = Z_{j-1} \quad \text{if } i = 1 \text{ or } i > k, \text{ and } 2 \leqslant j \leqslant k, \tag{7}$$

$$f_h(Z_i) = 0, \quad f_h(Z_{n_h+j}) = \sum_{t=1}^{j-1} Z_{n_h+t} + a_h Z_{n_h+j} \tag{7'}$$

$$\text{if } i \leqslant n_h \text{ or } i > n_{h+1}, \ 1 \leqslant j \leqslant l_h, \text{ and } 1 \leqslant h \leqslant r,$$

$$f_{h+r}(Z_i) = 0, \quad f_{h+r}(Z_{p_h+j}) = \sum_{t=1}^{\varepsilon_j} Z_{p_h+t} + (-1)^{j-1} d_h Z_{p_h+\delta_j} + e_h Z_{p_h+j} \tag{7''}$$

$$\text{if } i \leqslant p_h \text{ or } i > p_{h+1}, \ 1 \leqslant j \leqslant 2m_h, \text{ and } 1 \leqslant h \leqslant s.$$

The standard computation shows

$$f_0^k = 0, \quad (f_h - a_h I)^{l_h} = 0 \quad \text{and} \quad (f_i^2 + b_i f_i + c_i I)^{m_i} = 0$$

$$\text{for } h = 1, \ldots, r \text{ and } i = r+1, \ldots, r+s.$$

Denoting by T the distribution on M spanned by $Z_1, \ldots, Z_d$, we see that the tensor field $\bar{f} = f_0+\ldots+f_{r+s}$ satisfies the equation $Q(\bar{f}|T) = 0$ and that $(\bar{f}|T)^{d-1}(x), \ldots, (\bar{f}|T)(x), I$ are linearly independent for every x of M.

If T' is the distribution complementary to T and $\dim T'$ is even (odd), then for any irreducible polynomial Q' of degree $d' = 1, 2$ ($d' = 1$) there is a tensor field $\tilde{f}$ on M such that $\tilde{f}|T = 0$ and $Q'(\tilde{f}|T') = 0$ (the field $\tilde{f}$ can be defined by formulas analogous to (7)-(7'')). If Q' is a divisor of Q, then the tensor field $f = \bar{f}+\tilde{f}$ is a Q-structure on M.

COROLLARY 3.1. *If G is a Lie group, Q is a polynomial of degree $d \leqslant n = \dim G$, and n is even or Q has a real root, then there is a left-invariant Q-structure on G.*

COROLLARY 3.2. *If M is parallelizable and* $\dim M$ *is even (odd), then there is an almost complex structure (an f-structure of arbitrary rank $2r < \dim M$) on M.*

Now we return to the situation considered in Sections 1 and 2. M is a basis of a principal fibre bundle P with a group G and a connection Γ.

COROLLARY 3.3. *If M admits a Q-structure f of degree d and n is even or Q has a real root, then P admits a Q-structure F of rank $r \geqslant \operatorname{rank} f$.*

Proof. Let Q' be a divisor of Q such that there is a Q'-structure $\bar{f}$ on G. Put $F = f^h+\bar{f}^*$. Then

$$Q(F)(Z^h) = \big(Q(f)(Z)\big)^h = 0 \quad \text{and} \quad Q(F)(A^*) = \big(Q(\bar{f})(A)\big)^* = 0.$$

If $Q_0(F) = 0$ for some polynomial Q_0 of degree $d_0 < d$, then $Q_0(f)(Z) = \pi_*\big(Q_0(F)(Z^h)\big) = 0$ for every Z. Since this is not possible, F is a Q-structure. Moreover,

$$r = \operatorname{rank} F = \operatorname{rank} f + \operatorname{rank}\bar{f} \geqslant \operatorname{rank} f.$$

COROLLARY 3.4. *If M is an almost complex manifold and* $\dim G$ *(or* $\dim P$*) is even, then P admits an almost complex structure. In particular, P is orientable.*

COROLLARY 3.5. *If M is equipped with an f-structure of rank r, then P admits an f-structure of every rank r' such that $r \leqslant r' \leqslant r+2[n/2]$, where $n = \dim G$.*

4. Integrability. Let us consider a polynomial structure f on M with the structure polynomial

$$Q(x) = a_{m+1}x^{m+1}+\ldots+a_2x^2+x. \tag{8}$$

It defines two complementary distributions T_1 and T_2 with projectors

$$\pi_1 = -a_{m+1}f^m-\ldots-a_2f \quad \text{and} \quad \pi_2 = a_{m+1}f^m+\ldots+a_2f+I,$$

respectively. Clearly,

$$\pi_1+\pi_2 = I, \quad \pi_1\pi_2 = \pi_2\pi_1 = 0,$$

$$\pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_2 f = f\pi_2 = 0, \quad \pi_1 f = f\pi_1 = f.$$

Let f' be a left-invariant Q-structure on G, $\bar{f} = f^h + f'^*$. Denote by T_1', T_2' and π_1', π_2' ($\bar{T}_1, \bar{T}_2$ and $\bar{\pi}_1, \bar{\pi}_2$, respectively) distributions and projectors defined in analogous manner for f' (for $\bar{f}$, respectively). Of course, $X \in \bar{T}_i$ if and only if $\pi_* X \in T_i'$ and $\omega(X) \in T_i'$, $i = 1, 2$. Consequently, $\dim \bar{T}_i = \dim T_i + \dim T_i'$.

THEOREM 4.1. *Let a connection Γ be flat. Then the distribution $\bar{T}_1$ (or $\bar{T}_2$, respectively) is integrable if and only if the distributions T_1 and T_1' (or T_2 and T_2', respectively) are integrable.*

Proof. Repeating the proof of Ishihara and Yano [5], one can verify that the distribution $\bar{T}_1$ is integrable if and only if

$$(9) \qquad \bar{\pi}_2\big([\bar{f}, \bar{f}](\bar{\pi}_1 X, \bar{\pi}_1 Y)\big) = 0 \quad \text{for any } X, Y.$$

Of course, the analogous facts hold for T_1 and T_1'. If Γ is flat, then it follows from Theorem 1.1 and Proposition 2.2 that

$$\begin{aligned}\bar{\pi}_2\big([\bar{f}, \bar{f}](\bar{\pi}_1 Z^h, \bar{\pi}_1 Z'^h)\big) &= \bar{\pi}_2\big([\bar{f}, \bar{f}]\big((\pi_1 Z)^h, (\pi_1 Z')^h\big)\big) \\ &= \bar{\pi}_2\big(\big([f, f](\pi_1 Z, \pi_1 Z')\big)^h\big) = \big(\pi_2 [f, f](\pi_1 Z, \pi_1 Z')\big)\end{aligned}$$

and, similarly,

$$\bar{\pi}_2\big([\bar{f}, \bar{f}](\bar{\pi}_1 A^*, \bar{\pi}_1 B^*)\big) = \big(\pi_2' [f', f'](\pi_1' A, \pi_1' B)\big)^*.$$

Moreover,

$$(10) \qquad \begin{aligned}[\bar{f}, \bar{f}](A^*, Z^h) = [(f'A)^*, (fZ)^h] - \bar{f}[A^*, (fZ)^h] - \\ - \bar{f}[(f'A)^*, Z^h] + \bar{f}^2 [A^*, Z^h] = 0.\end{aligned}$$

Thus (9) holds if and only if

$$\pi_2 [f, f](\pi_1 Z, \pi_1 Z') = 0 \quad \text{and} \quad \pi_2' [f', f'](\pi_1' A, \pi_1' B) = 0$$

for every Z, Z' and A, B. This proves our assertion.

An analogous argument works for the distribution $\bar{T}_2$, since $\bar{T}_2$ is integrable if and only if

$$[\bar{f}, \bar{f}](\pi_2 X, \pi_2 Y) = 0 \quad \text{for every } X, Y.$$

We say that a Q-structure f is *partially integrable* (respectively, *integrable*) if $[f, f](\pi_1 X, \pi_1 Y) = 0$ for any X, Y (respectively, $[f, f] = 0$). It is known [3] that an f-structure is partially integrable if and only if the distribution T_1 is integrable and, for any integral manifold N of T_1, the almost complex structure $f|N$ on N is integrable. And an f-structure is integrable if and only if every point x of M has a neighbourhood U equipped with a coordinate system $(u^1, \ldots, u^m)$ such that the matrix (f_j^i), where $f_j^i = du^i(f(\partial/\partial u^j))$, has the form

$$\begin{pmatrix} 0 & -1_r & 0 \\ 1_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } 2r = \operatorname{rank} f.$$

THEOREM 4.2. *If Γ is flat, then $\bar{f}$ is partially integrable (respectively, integrable) if and only if both Q-structures f and f' are partially integrable (respectively, integrable).*

The theorem can be proved analogously to Theorem 4.1.

COROLLARY 4.1. *If M and G are complex manifolds, and a principal fibre bundle P over M with a structure group G admits a flat connection, then the manifold P can be equipped with a structure of a complex manifold.*

5. Framed and normal Q-structures. The polynomial structure f with the structural polynomial Q of form (8) is said to be *framed* if the distribution T_2 is framed, i.e. if there are linear independent vector fields $Z_1, \ldots, Z_r$, $r = \dim T_2$, spanning T_2, and r forms ω^i such that

$$\omega^i(Z_j) = \delta_j^i, \qquad \pi_2 = Z_i \otimes \omega^i. \tag{11}$$

If, in addition,

$$[f, f] + Z_i \otimes d\omega^i = 0$$

and the forms $d\omega^i$ are of bidegree $(1, 1)$ with respect to f, i.e. if

$$d\omega^i(Z, fZ') + d\omega^i(fZ, Z') = 0 \quad \text{for every } Z, Z',$$

then f is called a *normal Q-structure* (cf. [1], [3], [4] and [7]).

LEMMA 5.1. *Every left-invariant Q-structure f' on G is framed. More precisely, there are fields $A_1, \ldots, A_{r'} \in \mathfrak{g}$ and left-invariant forms $\theta^1, \ldots, \theta^{r'}$ $(r' = \dim T_2')$ such that*

$$\theta^i(A_j) = \delta_j^i, \qquad \pi_2' = A_j \otimes \theta^j. \tag{12}$$

Proof. Since the distribution T_2' is left invariant, i.e. $T_2'(g) = L_{g*}T_2'(e)$ for any $g \in G$, vector fields $A_1, \ldots, A_{r'}$ span T_2' if only $A_1(e), \ldots, A_{r'}(e)$ span $T_2'(e)$. Take vector fields $A_{r'+1}, \ldots, A_n$ such that $A_1, \ldots, A_n$ form a basis of $\mathscr{G}$. We can define left-invariant forms $\theta^1, \ldots, \theta^n$ by putting $\theta^k(A_l) = \delta_l^k$, $k, l \leqslant n$. Clearly,

$$\pi_2' = A_1 \otimes \theta^1 + \ldots + A_{r'} \otimes \theta^{r'}.$$

In the sequel, i, j and k, l run, respectively, from 1 through $r = \dim T_2$, from 1 through $r' = \dim T_2'$, and from 1 through $r + r' = \dim \bar{T}_2$.

THEOREM 5.1. *If f is framed, then $\bar{f} = f^h + f'^*$ is framed. If Γ is flat, and f and f' are normal, then $\bar{f}$ is normal.*

Proof. Let Z_i and ω^i (respectively, A_j and θ^j) satisfy (11) (respectively, (12)). Put $X_i = Z_i^h$, $X_{r+j} = A_j^*$, $\eta^i = \pi^*\omega^i$ and $\eta^{r+j} = \theta^j \circ \omega$. Of course, $X_1, \ldots, X_{r+r'}$ span $\overline{T}_2$ and $\eta^k(X_l) = \delta_l^k$. Besides,

$$\eta^{r+j}(Z^h) = \eta^i(A^*) = 0$$

and thus

$$\bar{\pi}_2(Z^h) = (\pi_2 Z)^h = \big(\omega^i(Z)Z_i\big)^h = \eta^i(Z^h)X_i = \eta^k(Z^h)X_k$$

and

$$\bar{\pi}_2(A^*) = (\pi_2' A)^* = \big(\theta^j(A)A_j\big)^* = \eta^{r+j}(A^*)X_{r+j} = \eta^k(A^*)X_k.$$

Therefore, $\bar{\pi}_2 = X_k \otimes \eta^k$, and $\overline{T}_2$ is framed.

Let us suppose that Γ is flat, and f and f' are normal. Then we have

$$d\eta^i(Z^h, \bar{f}Z'^h) + d\eta^i(\bar{f}Z^h, Z') = \big(d\omega^i(Z, fZ') + d\omega^i(fZ, Z')\big)\circ\pi = 0,$$

$$d\eta^i(A^*, \bar{f}X) + d\eta^i(\bar{f}A^*, X) = 0,$$

$$d\eta^{r+j}(Z^h, \bar{f}X) + d\eta^{r+j}(\bar{f}Z^h, X) = 0$$

and

$$d\eta^{r+j}(A^*, \bar{f}B^*) + d\eta^{r+j}(\bar{f}A^*, B^*) = d\theta^j(A, f'B) + d\theta^j(f'A, B) = 0.$$

Thus $d\eta^k$ are of bidegree $(1,1)$ with respect to $\bar{f}$. Finally,

$$\begin{aligned}([\bar{f},\bar{f}] + X_k\otimes d\eta^k)(Z^h, Z'^h) &= \big([f,f](Z,Z')\big)^h + \big(d\omega^i(Z,Z')\circ\pi\big)Z_i^h \\ &= \big(([f,f] + Z_i\otimes d\omega^i)(Z,Z')\big)^h = 0,\\ ([\bar{f},\bar{f}] + X_k\otimes d\eta^k)(A^*, B^*) &= \big([f',f'](A,B)\big)^* + d\theta^j(A,B)A_j^* \\ &= \big(([f',f'] + A_j\otimes d\theta^j)(A,B)\big)^* = 0\end{aligned}$$

and (cf. (10))

$$\begin{aligned}&([\bar{f},\bar{f}] + X_k\otimes d\eta^k)(Z^h, A^*)\\ &\qquad = Z^h(\theta^j A)\cdot X_{r+j} - A^*(\omega^i Z\circ\pi)\cdot X_i - \eta^k([Z^h, A^*])\cdot X_k = 0.\end{aligned}$$

Hence $\bar{f}$ is normal.

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