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# SOME PROPERTIES OF OPEN AND RELATED MAPPINGS

BY

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The purpose of this paper* is to provide solutions to a number of problems of A. Lelek, D. R. Read, J. Krasinkiewicz, and J. J. Charatonik concerning open and related mappings.

In Section 1 it is proved that if $f\colon X \to Y$ is an open mapping of metric continua, then $f$ can be extended to an open mapping $f^*\colon X^* \to Y^*$ of Peano continua such that $f^*(X^*\setminus X) = Y^*\setminus Y$.

Krasinkiewicz and Minc [8] and Oversteegen [19] have proved independently that there is a continuum $X$ which admits a monotone open retraction $r$ onto an arc $Y$ such that $r^{-1}(y)$ is a non-degenerate continuum for each $y \in Y$. In Section 2 of this paper, Oversteegen's construction is generalized in two directions.

We are very much indebted to Professor J. Krasinkiewicz whose remarks gave simplifications of some proofs of this paper.

**1. Extensions of open mappings.** All spaces in this paper are assumed to be metric. A *compactum* is a compact metric space and a *continuum* is a connected compactum. We will prove the following

THEOREM 1. *Let $f$ be an open continuous mapping of a continuum $X$ onto $Y$. Then there are locally connected continua $X^*$ and $Y^*$ such that $X$ and $Y$ are subsets of $X^*$ and $Y^* = f(X^*)$, respectively,* $\dim X = \dim X^*$, $\dim Y = \dim Y^*$, *and there is an extension $f^*$ of $f$ from $X^*$ to $Y^*$ such that $f^*$ is open and $f^*(X)$ does not intersect $f^*(X^*\setminus X)$.*

Proof. Let $Q$ denote the Hilbert cube with the metric

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i - y_i|,$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are points in $Q$. The projection onto the second $Q$ in the product $Q \times Q$ will be denoted by $\pi$, and the straight-line interval joining points $x, y \in Q \times Q$ by $[x, y]$. We assume that

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the metric in $Q \times Q$ is given by

$$d((x, y), (x', y')) = \varrho(x, x') + \varrho(y, y'),$$

where $(x, y)$, $(x', y') \in Q \times Q$, and that $X$ and $Y$ are contained in $\{0\} \times Q$, where $\{0\} = (0, 0, \ldots) \in Q$.

First, we will find a sequence of finite collections $\mathscr{G}_i$ of open subsets of $Y$ and finite collections $\mathscr{G}_i^G$ of open subsets of $X$ for each $G \in \mathscr{G}_i$ such that

(1) *$\mathscr{G}_i$ is an open finite cover of $Y$ with* $\operatorname{mesh} \mathscr{G}_i < 1/i$,

(2) *$\mathscr{G}'_i = \bigcup \{\mathscr{G}_i^G \colon G \in \mathscr{G}_i\}$ is an open finite cover of $X$ with* $\operatorname{mesh} \mathscr{G}'_i < 1/i$,

(3) *$f(H) = G$ for each $H \in \mathscr{G}_i^G$ and $G \in \mathscr{G}_i$,*

(4) *$f^{-1}(G) = \bigcup \{H \colon H \in \mathscr{G}_i^G\}$ for $G \in \mathscr{G}_i$.*

Since $f$ is continuous, there is a positive number $\delta$ such that $\delta < 1/2^i$, and if $d(x', x) < \delta$, then $d(f(x'), f(x)) < 1/2^i$. For a given point $y \in Y$ take an arbitrary finite cover $\{B(x_1, \delta), \ldots, B(x_n, \delta)\}$ of $f^{-1}(y)$ with $\{x_1, \ldots, x_n\} \subset f^{-1}(y)$, where $B(z, r)$ denotes the open ball with centre $z$ and radius $r$. Since the mapping $f$ is open, the set $\bigcap_{i=1}^{n} f(B(x_i, \delta))$ is open. The continuity of $f$ implies the existence of an open set $G$ with

$$G \subset \bigcap_{i=1}^{n} f(B(x_i, \delta))$$

such that $f^{-1}(y) \subset f^{-1}(G) \subset \bigcup_{i=1}^{n} B(x_i, \delta)$. Put

$$\mathscr{G}_i^G = \{B(x_i, \delta) \cap f^{-1}(G) \colon i = 1, 2, \ldots, n\}.$$

Then $y \in G$, $\operatorname{diam} G < 1/i$, $\operatorname{diam} H < 1/i$, $f(H) = G$ for $H \in \mathscr{G}_i^G$, and $f^{-1}(G) = \bigcup \{H \colon H \in \mathscr{G}_i^G\}$. Therefore, since $Y$ is compact, there is a finite collection $\mathscr{G}_i$ of sets $G$ which covers $Y$. The equalities

$$\bigcup \{H \in \mathscr{G}_i^G \colon G \in \mathscr{G}_i\} = \bigcup \{f^{-1}(G) \colon G \in \mathscr{G}_i\} = f^{-1}(\bigcup \{G \colon G \in \mathscr{G}_i\}) = f^{-1}(Y) = X$$

imply that all the conditions (1)-(4) are satisfied for $\mathscr{G}_i$ and $\mathscr{G}_i^G$ constructed in this way.

Moreover,

(5) *if $H \in \mathscr{G}_{i+1}$, $H' \in \mathscr{G}_{i+1}^H$, $G \in \mathscr{G}_i$, and $H \cap G \neq \emptyset$, then there is $G' \in \mathscr{G}_i^G$ such that $H' \cap G' \neq \emptyset$.*

Indeed, the relations

$$\emptyset \neq H \cap G = f(H') \cap G = f(H' \cap f^{-1}(G))$$

and

$$f^{-1}(G) = \bigcup \{G' \colon G' \in \mathscr{G}_i^G\}$$

imply $H' \cap G' \neq \emptyset$ for some $G' \in \mathscr{G}_i^G$.

Now, for each $G \in \mathscr{G}_i$ we choose an arbitrary point $y_i^G$ belonging to $G$ and for each $H \in \mathscr{G}_i^G$ we choose an arbitrary point $x_i^{H,G}$ in $H \cap f^{-1}(y_i^G)$. Further, if two distinct elements of $\mathscr{G}_i^G$ intersect, then we also choose a second point $z_i^G$ belonging to $G$.

We order the set $\{x_i^{H,G}: H \in \mathscr{G}_i^G, \mathscr{G} \in \mathscr{G}_i \text{ and } i = 1, 2, \ldots\}$ in a sequence $\alpha = \{a_1, a_2, \ldots\}$ in such a way that if $a_n = x_i^{H,G}$, then $i \leqslant n$. Similarly, the set $\{x: x = y_i^G \text{ or } z_i^G, G \in \mathscr{G}_i \text{ and } i = 1, 2, \ldots\}$ is ordered in a sequence $\beta = \{b_1, b_2, \ldots\}$ such that if $b_n = y_i^G$ or $z_i^G$, then $i \leqslant n$.

Now, let $\gamma = \{c_1, c_2, \ldots\}$ and $\gamma = \alpha$ or $\beta$. Put

$$\gamma(c_i) = (\delta_1^i, \delta_2^i, \ldots) \times \{\pi(c_i)\} \in Q \times Q,$$

where

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Observe that

(6) *the intervals* $[\gamma(c_i), \gamma(c_j)]$ *and* $[\gamma(c_m), \gamma(c_n)]$ *either coincide, or are disjoint or have only one end-point in common;*

(7) $d(c_i, \gamma(c_i)) < 1/2^i$ *for* $i = 1, 2, \ldots$

Put

$$X_n = \bigcup \{[\alpha(x_n^{H,G}), \alpha(x_n^{H',G'})]: H, H' \in \mathscr{G}_n' \text{ and } H \cap H' \neq \emptyset\},$$

$$Y_n = \bigcup \{[\beta(y_n^G), \beta(y_n^H)]: G, H \in \mathscr{G}_n \text{ and } G \cap H \neq \emptyset\} \cup$$
$$\cup \bigcup \{[\beta(y_n^G), \beta(z_n^G)]: G \in \mathscr{G}_n \text{ and } z_n^G \text{ is defined}\},$$

$$X_{n+1,n} = \bigcup \{[\alpha(x_n^{H,G}), \alpha(x_{n+1}^{H',G'})]: H \in \mathscr{G}_n', H' \in \mathscr{G}_{n+1}', \text{ and } H \cap H' \neq \emptyset\},$$

and

$$Y_{n+1,n} = \bigcup \{[\beta(y_n^G), \beta(y_{n+1}^H)]: G \in \mathscr{G}_n, H \in \mathscr{G}_{n+1}, \text{ and } H \cap G \neq \emptyset\}.$$

Finally, we write

$$X^* = X \cup \bigcup_{n=1}^{\infty} (X_n \cup X_{n+1,n}) \quad \text{and} \quad Y^* = Y \cup \bigcup_{n=1}^{\infty} (Y_n \cup Y_{n+1,n}).$$

We define the mapping $f^*$ from $X^*$ onto $Y^*$ putting $f^*(x) = f(x)$ for $x \in X$, $f^*(\alpha(x_n^{H,G})) = \beta(y_n^G)$ for each $H \in \mathscr{G}_n^G$, $G \in \mathscr{G}_n$ and $n = 1, 2, \ldots$ if $H, H' \in \mathscr{G}_n^G$ are distinct but have non-void intersection, the middle point of the interval $[\alpha(x_n^{H,G}), \alpha(x_n^{H',G})]$ is mapped by $f^*$ onto $\beta(z_n^G)$, and $f^*$ is linear on the rest of $X^*$.

In this way the construction is complete. Omitting the details in easy cases we are going now to show that $X^*$, $Y^*$, and $f^*$ satisfy the theorem. It follows from conditions (1)-(6) that if

$$X^m = \bigcup_{n=1}^{m} (X_n \cup X_{n+1,n}) \quad \text{and} \quad Y^m = \bigcup_{n=1}^{m} (Y_n \cup Y_{n+1,n}),$$

then

(8) *$X^m$ and $Y^m$ are locally connected continua,*

(9) *$f^*|X^m$ is an open continuous mapping from $X^m$ onto $Y^m$.*

Moreover, by (7),

(10) *the diameter of each straight-line interval contained either in $X_n \cup X_{n+1,n}$ or in $Y_n \cup Y_{n+1,n}$ is not greater than $(3\cdot 2^{n-1})^{-1}$.*

Conditions (7), (8), and (10) imply that

(11) *$X^*$ and $Y^*$ are continua.*

Now, if $\lim_{m\to\infty} x_m = x \in X$ and $x_m = \alpha(x_m^{H,G})$ for some $G \in \mathscr{G}_m$ and $H \in \mathscr{G}_m^G$, then

$$\lim_{m\to\infty} x_m^{H,G} = x$$

by (7). Since $f(x) = \lim_{m\to\infty} f(x_m^{H,G}) = \lim_{m\to\infty} y_m^G$, $f^*(x_m) = \beta(y_m^G)$ for $m = 1, 2, \ldots,$ and $\lim_{m\to\infty} \beta(y_m^G) = \lim_{m\to\infty} y_m^G$ (by (7)), we conclude that

$$\lim_{m\to\infty} f^*(x_m) = f(x) = f^*(x).$$

Therefore, by (10), we have

(12) *$f^*$ is continuous.*

If $\lim_{m\to\infty} y_m = y \in Y$, $y_m = \beta(y_m^G)$ for some $G \in \mathscr{G}_m$, then

$$\lim_{m\to\infty} y_m^G = y.$$

Since $f$ is open for an arbitrary $x \in f^{-1}(y)$, there is a sequence $x_m$ in $X$ such that $\lim_{m\to\infty} x_m = x$ and $f(x_m) = y_m^G$. But $x_m$ belongs to some $H \in \mathscr{G}_m^G$. Since

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} x_m^{H,G} = \lim_{m\to\infty} \alpha(x_m^{H,G})$$

and

$$\alpha(x_m^{H,G}) \in (f^*)^{-1}(\beta(y_m^G)) = (f^*)^{-1}(y_m),$$

we conclude, by (9), that

(13) *$f^*$ is open.*

It is clear that

(14) $\dim X = \dim X^*$, $\dim Y = \dim Y^*$, *and* $f^*(X) \cap f^*(X^* \setminus X) = \emptyset$.

It remains to prove that

(15) *$X$ is locally connected.*

Fix $A \in \mathscr{G}_n^G$ and put

$$A_0 = \{\alpha(x_n^{A,G})\},$$

$$A_k = \bigcup \{[\alpha(x_{n+k}^{H,G}),\ \alpha(x_{n+k}^{H',G'})]:$$
$$H,\ H' \in \mathcal{G}'_{n+k},\ H \cap H' \neq \emptyset,\ H \cap A \neq \emptyset \neq H' \cap A\},$$
$$A_{0,1} = \bigcup \{[\alpha(x_n^{A,G}),\ \alpha(x_{n+1}^{H',G'})]:\ H' \in \mathcal{G}'_{n+1},\ H' \cap A \neq \emptyset\},$$
$$A_{k,k+1} = \bigcup \{[\alpha(x_{n+k}^{H,G}),\ \alpha(x_{n+k+1}^{H',G'})]:$$
$$H \in \mathcal{G}_{n+k},\ H' \in \mathcal{G}_{n+k+1},\ A \cap H' \neq \emptyset \neq H' \cap A \text{ and } H \cap H' \neq \emptyset\}$$
$$\text{for } k = 1, 2, \ldots$$

If

$$A' = \bigcup_{k=0}^{\infty} (A_k \cup A_{k,k+1}),$$

then, as above (because if $H' \in \mathcal{G}_{n+k+1}$, $H' \cap A \neq \emptyset$, then there is $H \in \mathcal{G}_{n+k}$ such that $A \cap H \neq \emptyset \neq H \cap H'$),

(16) *$A'$ is connected.*

If $A$ has a small diameter, then $A'$ has also a small diameter. Since the closure of $A'$ contains $A$, we conclude that $X$ is locally connected by (16).

The proof of Theorem 1 is complete.

By a *curve* is meant 1-dimensional continuum.

From Theorem 1 we obtain

THEOREM 2. *Let $f$ be an open continuous mapping of a curve $X$ onto $Y$. Then there exist continua $X^*$ and $Y^*$ such that $X \subset X^*$, $Y \subset Y^*$ and an extension $f^*\colon X^* \to Y^*$ of $f$ such that $f^*(X^*\setminus X) = Y^*\setminus Y$ and such that $X^*$ is the Menger universal curve.*

Before beginning the proof of Theorem 2, we will recall some definitions and facts. If $p$ is a point of a continuum $K$, then $p$ is said to be a *local cut point* of $K$ if, for some connected open set $D$ of $K$, $D\setminus\{p\}$ is not connected.

The *Sierpiński universal curve* $S$ (see [9], p. 275) is a plane nowhere dense locally connected continuum which is obtained as follows: Partition the unit square $[0, 1] \times [0, 1]$ into nine congruent squares and delete the interior of the central square. In a similar way, partition each of the remaining eight squares and delete the central square from each of them. This process is continued inductively. The points of the unit square which are not deleted constitute the continuum $S$.

The *Menger universal curve* $U$ is the continuum consisting of all points $(x, y, z)$ in the cube $[0, 1] \times [0, 1] \times [0, 1]$ such that $(x, y)$, $(x, z)$, $(y, z) \in S$.

We shall need the following theorem of Anderson (see [1], Theorem XII, p. 13):

THEOREM. *Let $K$ be a one-dimensional locally connected continuum. In order that $K$ be homeomorphic to $U$ it is necessary and sufficient that $K$*

*contain no open subset imbeddable in the plane and $K$ contain no local cut point.*

Proof of Theorem 2. First, in the same way as in the proof of Theorem 1 we construct Peano continua $X_0^*$ and $Y_0^*$ such that $X \subset X_0^*$, $Y \subset Y_0^*$ and an open mapping $f_0^*$ from $X_0^*$ onto $Y_0^*$ such that $f_0^*$ is an extension of $f$ and $f_0^*(X_0^*\backslash X) \subset Y_0^*\backslash Y$.

To obtain $X^*$, $Y^*$, and $f^*$ we will replace each interval contained in $X_0^*\backslash X$ and $Y_0^*\backslash Y$ by a copy of Menger's curve. We can do it as follows: For each maximal interval $L$ lying in $X_0^*\backslash X$ (or in $Y_0^*\backslash Y$) we take a linear mapping $h_L$ from $L$ onto the interval $[0, 1]$ or onto the interval $[0, 2]$. The second case is for intervals $L \subset X^*\backslash X$ of the form $[\alpha(x_n^{H,G}), \alpha(x_n^{H',G})]$ with $H \cap H' \neq \emptyset$. Define $h'_L$: $L \times I^2 \to h_L(L) \times I^2$ by

$$h'_L(r, t, s) = (h_L(r), t, s)$$

and put

$$V = \{(r, t, s) \in R^3 \colon 0 \leqslant r \leqslant 2,\ 0 \leqslant t, s \leqslant 1,\ (r, t, s) \in U \text{ or } (r-1, t, s) \in U\}.$$

Consider a set $X_1^*$ in $Q \times Q \times I^2$ defined by

$$X_1^* = \bigcup \{(h'_L)^{-1}(V) \colon L \text{ is a maximal interval in } X_0^*\backslash X\} \cup X \times I^2$$

and an equivalence relation $\varrho_X$ in $X_1^*$ defined by $(r, t, s)\, \varrho_X (r', t', s')$ if and only if either $(r, t, s) = (r', t', s')$ or $r = r' \in X$, and let $\varphi_X$ be the natural projection from $X_1^*$ onto $X^* = X_1^*/\varrho_X$. In a similar way we define $Y_1^*$, $\varrho_Y$, $\varphi_Y$, and $Y^* = Y_1^*/\varrho_Y$. The mapping $f^*$: $X^* \to Y^*$ is defined as follows: $f^*(x) = \varphi_Y(f^*(r), t, s)$ provided $(r, t, s) \in \varphi_X^{-1}(x)$ and $x \in X^*$. An easy computation shows that $X^*$, $Y^*$, and $f^*$ satisfy Theorem 2 ($X^*$ does not contain local cu points by (16)).

In Problem 20 of the University of Houston Problem Brook (UHPB), A. Lelek has asked about just such implications as we have proved in Theorems 1 and 2.

The following question of Lelek remains open: Can $X^*$ in Theorem 1 be the Hilbert cube?

For other classes of mappings we have the following observations:

THEOREM 3. *Suppose $f$ is a monotone mapping of a continuum $X$ onto $Y$ and $X \subset Q$. Then there is a monotone mapping $f^*$: $Q \to f^*(Q)$ such that $f^*|X = f$ and $f^*(X) \cap f^*(Q\backslash X) = \emptyset$.*

In fact, the mapping $f^*$ such that $f^*|X = f$ and $f^*|(Q\backslash X)$ is the identity has such properties.

Recall that a mapping $f$: $X \to Y$ is *confluent* (resp. *weakly confluent*) if for each continuum $K \subset Y$ each component (resp. some component) of $f^{-1}(K)$ is mapped by $f$ onto $K$.

Problem 20 has a negative answer for the class of confluent maps and for the class of weakly confluent maps.

It is known that there is a confluent mapping $f$ from a continuum $X$ onto $Y$ such that $f \times \mathrm{id}_{[0,1]}$ is not confluent (see [15]). This mapping cannot be extended to a confluent mapping $f^*$ defined on a locally connected continuum $X^*$ with $X \subset X^*$ and $f^*(X^*\backslash X) \cap f^*(X) = \emptyset$. For suppose that there is such an extension $f^*$. Then $f^*$ is a composition of a monotone mapping and an open mapping (see [13]); thus $f^* \times \mathrm{id}_{[0,1]}$ is also a composition of a monotone mapping and an open mapping. Since

$$X \times [0, 1] = (f^* \times \mathrm{id}_{[0,1]})^{-1}(f^*(X) \times [0, 1]),$$

we infer $f^*|(X \times [0, 1])$ is confluent, a contradiction.

A stronger example is the following:

Example 1. Let $P$ be the standard $\sin 1/x$ curve in the plane

$$P = \{(x, \sin 1/x)\colon\ 0 < x \leqslant 1\} \cup \{0\} \times [-1, 1].$$

Let $r$ and $s$ be the points $(0, -1)$ and $(0, 1)$, respectively. Let

$$X = \{r\} \times [0, 1] \cup P \times \{0, 1\}$$

and define an equivalence relation $\sim$ on $X$ by setting $(x, y, t) \sim (x', y', t')$ if and only if $(x, y, t) = (x', y', t')$ or $x = x' = 0$ and $y = y'$, and denote by $f$ the canonical mapping from $X$ onto the quotient space $Y = X|\sim$. Then $f$ is confluent, and hence weakly confluent.

Suppose that there is a weakly confluent mapping $f^*\colon X^* \to Y^*$ defined as a locally connected continuum $X^*$ with $X \subset X^*$, $f^*|X = f$, $Y \subset Y^*$, and $f^*(X^*\backslash X) = Y^*\backslash Y$. Then $Y = f^*(X^*)$ is locally connected. There exist two sequences $\{s_n^0\}$ and $\{s_n^1\}$ converging to $f((s, 0))$ in $Y$ such that for each $n$ we have

$$s_n^0 \in f(P \times \{0\}) \backslash f(\{0\} \times [-1, 1] \times \{0\})$$

and

$$s_n^1 \in f(P \times \{1\}) \backslash f(\{1\} \times [-1, 1] \times \{0\}).$$

Since $Y^*$ is locally connected, there exist arcs $s_n^0 s_n^1$ joining points $s_n^0$ and $s_n^1$ and converging to the point $f((s, 0))$. Since $f^*$ is weakly confluent, there exist continua $\{C_n\}$ in $X^*$ such that $f^*(C_n) = s_n^0 s_n^1$. For $i = 0, 1$ and $n = 1, 2, \ldots$ the only point $x_n^i$ of $X^*$ which is mapped by $f^*$ onto $s_n^i$ is contained in $P\backslash(\{0\} \times [-1, 1] \times \{i\})$. Thus $C_n \cap [P \times \{i\}] \neq \emptyset$ for $i = 0, 1$ and $n = 1, 2, \ldots$ Since $f^*$ is continuous, we have

$$\operatorname*{Ls}_{n \to \infty} C_n \subset (f^*)^{-1} f((s, 0)) = f^{-1}(f((s, 0))).$$

The set $\operatorname{Ls}_{n\to\infty} C_n$ contains a continuum which intersects both $P\times\{0\}$ and $P\times\{1\}$, a contradiction, because $f^{-1}\big(f((s, 0))\big)$ is a two-point set.

**2. Monotone and continuous decompositions.**

THEOREM 4. *Let $N$ be an arbitrary continuum and let $t_0\in N$. Then there are a continuum $M$ containing $[0, 1]$, an open monotone retraction $\omega$ of $M$ onto $[0, 1]$, and a continuous mapping $\mu$ from $M$ onto $N$ such that*

(i) *$\omega^{-1}(t)$ is non-degenerate for each $t\in[0, 1]$;*

(ii) $\mu^{-1}(t_0) = [0, 1]$;

(iii) *if $m$ is a point in $M\setminus[0, 1]$, then there are closed-open sets $V_i$ in $M\setminus[0, 1]$ such that $\bigcap_{i=1}^{\infty}\omega(V_i) = \omega(m)$ and $m\in V_i$ for $i = 1, 2, \ldots$;*

(iv) *if $F\subset M\setminus[0, 1]$ is a continuum, then $\mu|F$ is a homeomorphism.*

Proof. The proof is based on the construction of Example 3.2 in [19], p. 118 (cf. [8], Example (3.3)). We use the notation from this example.

Let $\pi_1\colon [0, 1]\times[0, 1]\to[0, 1]$ be defined by $\pi_1(x, y) = x$. There is a compact set $Q\subset[0, 1]\times[0, 1]$ satisfying the following conditions:

(a) $C = \pi_1(Q)\subset[0, 1]$ is a Cantor set;

(b) $Q = \bigcup\{l_c\colon c\in C\}$, where $l_c\subset\pi_1^{-1}(c)$ is a – possibly degenerate – line segment with one end-point, the point $c$, on the $X$-axis;

(c) $l_c\neq\{c\}$ for some $c\in C$;

(d) if $l_c\neq\{c\}$, then there exist two sequences $\{a_i\}$ and $\{b_i\}$ in $C$ such that $a_i < a_{i+1} < c < b_{i+1} < b_i$ $(i = 1, 2, \ldots)$ and $\lim l_{a_i} = l_c = \lim l_{b_i}$.

Let $Q^* = Q\times R\subset R^3$ and let $P\subset\{0\}\times[0, 1]\times[0, 1]$ be the following continuum: Let $l\subset R^3$ be the closed line segment joining the points $(0, 1, 0)$ and $(0, 1, 1)$, and let $K\subset l$ be the Cantor ternary set. Hence, the $z$-coordinate of each point $x\in K$ has a unique ternary expansion $\sum_{n=1}^{\infty} x_n/3^n$, where $x_n\in\{0, 2\}$. Join by a straight-line segment each point $x\in K$, whose $z$-coordinate can be written as $\sum_{n=0}^{\infty} x_n/3^n$ $(x_n\in\{0, 2\})$, with the point $(0, 0, \sum_{n=1}^{\infty} y_n/2^n)$, where $y_n = 0$ if $x_n = 0$ and $y_n = 1$ if $x_n = 2$, and let $P$ be the resulting continuum.

For each point $t\in P$, $t = (0, y, z)$, let $A_t$ denote the straight closed line segment joining the point $t$ and the point $(1, y, z^2)$. Put $P^* = \bigcup\{A_t\colon t\in P\}$, $X^* = P^*\cap Q^*$, and $P_c = P^*\cap(l_c\times R)$, $c\in C$. Let

$$Y^* = \{(g, w, s)\in X^*\colon w = 0\}.$$

For each $x\in X^*$, there exists a – possibly degenerate – unique irreducible line segment $T_x$ joining $x$ and $Y^*$.

Define $r^*\colon X^*\to Y^*$ by $r^*(x) = T_x\cap Y^*$; then $r^*$ is a well-defined monotone retraction. Put

$$B_s = \{(q, 0, s) \in Y^*: 0 \leqslant q \leqslant 1\}, \quad 0 \leqslant s \leqslant 1,$$

and define an equivalence relation $R$ on $X^*$ by

$$xRy \Leftrightarrow x = y \text{ or } x, y \in B_z \quad \text{for some } z \in [0, 1].$$

Put $X = X^*/R$ and let $p$: $X^* \to X$ denote the quotient map. Put $Y = p(Y^*)$ and $p(B_z) = \{z\}$; then $Y \approx [0, 1]$. Since $p$ is a quotient map and $g = p \circ r^*$ is constant on each set $p^{-1}(x)$ for $x \in X$, $r^*$ induces a continuous map $r$: $X \to Y$.

To each $t \in [0, 1]$ we can assign a subcontinuum $N_t$ of $N$ (see [18], p. 65) such that

(1) $N_0 = \{t_0\}$, $N_1 = N$, $N_t \subsetneqq N_{t'}$ *if* $t < t'$ *and* $t_n \to t \Leftrightarrow N_{t_n} \to N_t$.

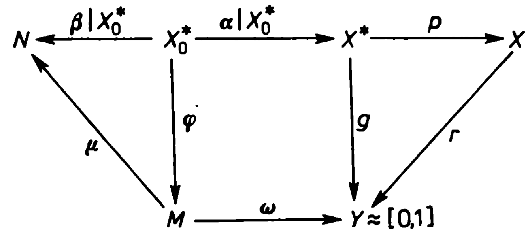
Let $\alpha$: $X^* \times N \to X^*$ and $\beta$: $X^* \times N \to N$ denote the natural projections from the Cartesian product $X^* \times N$ onto $X^*$ and $N$, respectively. Consider the set

$$X_0^* = \{(x, t) \in X^* \times N: x = (q, w, s) \text{ and } t \in N_w\}$$

and define an equivalence relation $R_0$ on $X_0^*$ by

$$xR_0y \Leftrightarrow \beta(x) = \beta(y) = t_0 \text{ and } r^*(\alpha(x)), r^*(\alpha(y)) \in B_z \text{ for some } z \in [0, 1]$$
$$\text{or } \beta(x) = \beta(y) \text{ and } \alpha(x), \alpha(y) \in T_z \text{ for some } z \in X^*.$$

Put $M = X_0^*/R_0$ and let $\varphi$: $X_0^* \to M$ denote the quotient map. Define $\omega(t) = g\alpha\varphi^{-1}(t)$ and $\mu(t) = \beta\varphi^{-1}(t)$ for $t \in M$. Since, if $x$, $y \in X_0^*$ and either $r^*(\alpha(x))$, $r^*(\alpha(y)) \in B_z$ or $\alpha(x)$, $\alpha(y) \in T_z$, then $g(\alpha(x)) = g(\alpha(y))$, we infer that $\omega$ is a well-defined single-valued mapping. Similarly, $\mu$ is well defined. In this way we obtain the following commutative diagram:




If $x \in X^*$, then we put

$$M_x = \{(x, t) \in X^* \times N: x = (g, w, s) \text{ and } t \in N_w\}.$$

It is obvious that $M_x \subset X_0^*$ and $M_x$ is homeomorphic to $N_w$. If $A \subset X^*$, let $M_A = \bigcup \{M_x \mid x \in A\}$.

(2) *If* $A \subset X^*$ *is a continuum, then* $M_A$ *is a continuum.*

Indeed, $M_x \to x$ gives a monotone mapping from $M_A$ onto $A$. Since $M = \varphi(X_0^*)$ and $X_0^* = M_{X^*}$, by (2) we have

(3) $M$ *is a metric continuum.*

Since $\alpha(M_{T_x}) = T_x$ and $g(T_x)$ is a point for $x \in X^*$, we obtain

(4) *$\omega\varphi(M_{T_x})$ is a point for $x \in X$.*

Consequently,

(5) *the mapping $\omega$ is monotone.*

In fact, if $(x, t)$ and $(x', t')$ are points of $X_0^*$ such that $\omega\varphi(x, t) = \omega\varphi(x', t')$, then $(r^*(x), t_0) \in M_{T_x}$, $(r^*(x'), t_0) \in M_{T_{x'}}$, and $\omega\varphi(r^*(x), t_0) = \omega\varphi(r^*(x'), t_0)$ by (4). Since $\varphi(r^*(x), t_0) = \varphi(r^*(x'), t_0)$, we infer that $\varphi(M_{T_x} \cup M_{T_{x'}})$ is connected by (2) and it is contained in $\omega^{-1}\omega\varphi(x, t)$. This implies that $\omega^{-1}(y)$ is connected for $y \in Y$. Now,

(6) *$\omega_1 = \omega \mid \varphi(Y^* \times \{t_0\})$ is a homeomorphism from $\varphi(Y^* \times \{t_0\})$ onto $Y$.*

If $y = (q, w, s) \in Y^*$, $y' = (q', w', s') \in Y^*$, and $g\alpha(y, t_0) = g\alpha(y', t_0)$, then $w = w' = 0$ and $s = s'$. Therefore, $(y, t_0) R_0 (y', t_0)$, which implies that $\omega_1$ is one-to-one.

Now we will prove

(7) *the mapping $\omega$ is open.*

It suffices to show that for each point $z \in Y$, each sequence $\{z_n\}$ in $Y$ converging to $z$, and each $y \in \omega^{-1}(z)$ there are $y_n \in \omega^{-1}(z_n)$ such that $\lim y_n = y$. Using (6) we can assume that $y \in \omega^{-1}(z) \backslash \varphi(Y^* \times \{t_0\})$ and let $(x, t) \in \varphi^{-1}(y)$. If $t = t_0$, then we have $\varphi(x, t) \in \varphi(Y^* \times \{t_0\})$ because $(r^*(x), t_0) R_0 (x, t_0)$ and $(r^*(x), t_0) \in Y^* \times \{t\}$. This contradiction shows that $t \neq t_0$. Since $(x, t) \in X_0^*$, we obtain $\alpha(x, t) = x \notin Y^*$. It follows from the proof of the openness of $r$ in Example (3.2) in [19], p. 121, that there are $x_n \in g^{-1}(z_n)$, $x_n = (q_n, w_n, s_n)$, such that $x_n \to x$. In particular, if $x = (q, w, s)$, then $w_n \to w$. Since $t \in N_w$, $N_{w_n} \to N_w$ (cf. (1)), we conlude that there are $t_n \in N_{w_n}$ such that $t_n \to t$. Then $(x_n, t_n) \to (x, t)$, $g\alpha(x_n, t_n) = z_n$. Thus $\varphi(x_n, t_n) \to \varphi(x, t) = y$ and $y_n = \varphi(x_n, t_n) \in \omega^{-1}(z_n)$, which completes the proof of (7).

(8) *$\omega^{-1}(y)$ is non-degenerate for each $y \in Y$.*

In fact, if $y \in Y$, then there is $x = (q, w, s) \in X^* \backslash Y^*$ such that $g(x) = y$. Therefore, $w \neq 0$ and there is $t \in N_w \backslash \{t_0\}$. Since $(q, w, s, t) R_0 (q', 0, s', t_0)$ and $t \neq t_0$ implies $w = 0$, we infer that $\varphi(x, t) \in \omega^{-1}(y) \backslash \omega_1^{-1}(y)$, i.e., $\omega^{-1}(y)$ is non-degenerate.

Since $(x, t_0) R_0 (r^*(x), t_0)$, we obtain $\varphi(x, t_0) \in \omega_1^{-1}(Y)$. Thus

(9) $\mu^{-1}(t_0) = \omega_1^{-1}(Y)$.

It remains to prove property (iii). If $x \in X^*$ and $t \in N$, then we put

$$M_t^x = \{(y, t) \in X_0^*: y \in T_x\}.$$

(10) *The set $M_t^x$ is a continuum.*

Indeed, if $w_0 = \min\{w \in [0, 1]: t \in N_w\}$, then $M_t^x$ is homeomorphic to the set $\{y \in T_x: y = (q, w, s) \text{ and } w \geqslant w_0\}$ which is an arc or a point.

(11) *If $m \in M \backslash \omega_1^{-1}(Y)$, then $\varphi^{-1}(m) = M_t^x$ for some $x \in X^*$, $t > 0$, and $\alpha\varphi^{-1}(m) \subset X^* \backslash Y^*$.*

Indeed, if $\varphi(y, t) = \varphi(y', t') = m$, then $t = t' > 0$ and $y, y' \in T_x \backslash Y^*$ for some $x \in X^*$ by the definition of the relation $R_0$ ($\varphi(y, t_0) \in \omega_1^{-1}(Y)$ for each $y$).

Now, assume that $(x, t) \in \varphi^{-1}(M \backslash \omega_1^{-1}(Y))$. Then we have $x \in X^* \backslash Y^*$ and $\alpha(\varphi^{-1}(M \backslash \omega_1^{-1}(Y))) \subset X^* \backslash Y^*$ by (11). The construction of $X^*$ implies that there are closed-open sets $W_i'$ in $X^* \backslash Y^*$ such that

$$\bigcap_{i=1}^{\infty} g(W_i') = g(x) \quad \text{and} \quad x \in W_i' \quad \text{for } i = 1, 2, \ldots$$

Sets $W_i = \alpha^{-1}(W_i') \cap \varphi^{-1}(M \backslash \omega_1^{-1}(Y))$ are closed-open in $M \backslash \omega_1^{-1}(Y)$ because $\varphi^{-1}(M \backslash \omega_1^{-1}(y)) \subset \alpha^{-1}(X^* \backslash Y^*)$. From (10) and (11) we conclude that $\varphi^{-1}\varphi(W_i) = W_i$. Therefore, $V_i = \varphi(W_i)$ are closed-open in $M \backslash \omega_1^{-1}(Y)$. Since

$$\bigcap_{i=1}^{\infty} \omega(V_i) = \bigcap_{i=1}^{\infty} g\alpha\varphi^{-1}(V_i) = \bigcap_{i=1}^{\infty} g\alpha(W_i) \subset \bigcap_{i=1}^{\infty} g(W_i') = g(x),$$

we obtain

$$\bigcap_{i=1}^{\infty} \omega(V_i) = g(x) = \omega\varphi(x, t),$$

i.e., condition (iii) is satisfied.

If $F \subset M \backslash [0, 1]$ is a continuum, then condition (iii) implies that $\omega(F)$ is a point. Moreover, if $(x, t), (x', t) \in \varphi^{-1}(F)$, then $x, x' \in T_{x_0}$ for some $x_0 \in X^*$, because $\varphi^{-1}(F)$ is a continuum by (10) and (11). Therefore $(x, t) R_0 (x', t)$. It follows that $\mu | F$ is one-to-one, which completes the proof of Theorem 4.

One can observe that small modifications of the proofs of Lemma 3, Lemma 4, and Theorem 1 in [17] give the following

LEMMA. *Every curve $X$ is the inverse limit of an inverse sequence $\{P_i, \pi_{i,j}\}$ with one-dimensional polyhedra $P_i$ and with bonding maps $\pi_{i,j}\colon P_j \to P_i$ such that $\pi_{i,j}^{-1}(t)$ is finite for all $t \in P_i$ and $i, j = 1, 2, \ldots$*

THEOREM 5. *Let $N$ be an arbitrary continuum, $t_0 \in N$, and let $X$ be a curve. Then there are a continuum $Y$ containing $X$, a monotone open retraction $r$ from $Y$ onto $X$, and a continuous mapping $h$ from $Y$ onto $N$ such that*

(i) *$r^{-1}(x)$ is non-degenerate for each $x \in X$;*

(ii) *$h^{-1}(t_0) = X$;*

(iii) *if $F$ is a quasi-component of $Y \backslash X$, then $F \subset r^{-1}(x)$ for some $x \in X$;*

(iv) *if $F \subset Y \backslash X$ is a continuum, then $h | F$ is a homeomorphism.*

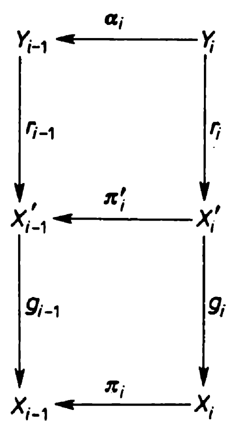
Proof. It follows from the Lemma that $X$ is the inverse limit of an inverse sequence $\{X_i, \pi_{i,j}\}$ with bonding maps $\pi_{i,j}$ such that point inverses under $\pi_{i,j}$ are finite and with one-dimensional polyhedra $X_i$. Put $\pi_{i,i-1} = \pi_i$ for brevity. Of course, we can assume that $X_1 = [0, 1]$. We will define, by induction on $i$, the following sequences: continua $X_i'$, continua $Y_i$, maps $\pi_{i+1}'\colon X_{i+1}' \to X_i$, maps $g_i\colon X_i' \to X_i$, maps $r_i\colon Y_i \to X_i'$, and maps $\alpha_{i+1}\colon Y_{i+1} \to Y_i$; all this in such a manner that

(1) *$g_i$ is a homeomorphism from $X'_i$ onto $X_i$;*

(2) *$X'_i \subset Y_i$ and $r_i$ is an open monotone retraction from $Y_i$ onto $X'_i$;*

(3) *if $y$ is a point in $Y_i \backslash X'_i$, then there are closed-open sets $V_n$ in $Y_i \backslash X'_i$ such that $\bigcap_{n=1}^{\infty} r_i(V_n) = r_i(y)$ and $y \in V_n$ for $n = 1, 2, \ldots$*

(4) *the diagram*




*is exactly commutative* (for definition see [20]);

(5) *$\alpha_i | r_i^{-1}(t)$ is a homeomorphism from $r_i^{-1}(t)$ onto $r_{i-1}^{-1}(\pi'_i(t))$ for each $t \in X'_i$.*

It follows from Theorem 4 that there are a continuum $M$ containing $[0, 1]$ and mappings $\mu$ and $\omega$ satisfying (i)-(iv) of Theorem 4. Put $Y_1 = M$, $X_1 = X'_1 = [0, 1]$, $r_1 = \omega$, and $g_1 = \mathrm{id}_{[0,1]}$. Then conditions (1)-(3) are satisfied. Moreover,

(6) *$\mu$ is a mapping from $Y_1$ onto $N$ such that $\mu^{-1}(t_0) = X_1$ and if $F \subset Y_1 \backslash X'_1$ is a continuum, then $\mu | F$ is a homeomorphism.*

Now assume that we have already defined $Y_i$, $X'_i$, $r_i$, $g_i$, $\alpha_i$, $\pi'_i$ for all $i \leqslant k$ in accordance with (1)-(5) for $i \leqslant k$. Let $\beta$ and $\gamma$ denote the projections from $X_{k+1} \times Y_k$ onto $X_{k+1}$ and $Y_k$, respectively. Put

$$Y_{k+1} = \{(x, y) \in X_{k+1} \times Y_k \colon \pi_{k+1}(x) = g_k r_k(y)\},$$

$$X'_{k+1} = \{(x, y) \in Y_{k+1} \colon y \in X'_k\},$$

$$g_{k+1}(t) = \beta(t) \text{ for } t \in X'_{k+1}, \qquad \pi'_{k+1}(t) = \gamma(t) \text{ for } t \in X'_{k+1},$$

$$\alpha_{k+1}(t) = \gamma(t) \text{ for } t \in Y_{k+1}, \qquad r_{k+1}(t) = g_{k+1}^{-1}\beta(t) \text{ for } t \in Y_{k+1}.$$

Since $X'_{k+1} = \{(x, y) \in X_{k+1} \times X'_k \colon y = g_k^{-1}\pi_{k+1}(x)\}$, we infer that $X'_{k+1}$ is a graph of $g_k^{-1}\pi_{k+1}$. Therefore,

(7) *$g_{k+1}$ is a homeomorphism from $X'_{k+1}$ onto $X_{k+1}$.*

Since $r_{k+1} | X'_{k+1} = (\beta | X'_{k+1})^{-1}(\beta | X'_{k+1})$, we conclude that

(8) *$r_{k+1}$ is a retraction from $Y_{k+1}$ onto $X'_{k+1}$.*

Since

$$r_{k+1}^{-1}(x, y) = \{(x, y')\in X_{k+1} \times Y_{k+1}\colon\ y'\in r_k^{-1} g_k^{-1} \pi_{k+1}(x)\},$$

we infer that

(9) $\alpha_{k+1} | r_{k+1}^{-1}(x, y)$ *is a homeomorphism from* $r_{k+1}^{-1}(x, y)$ *onto* $r_k^{-1} g_k^{-1} \pi_{k+1}(x)$ *for each* $(x, y)\in X'_{k+1}$.

Conditions (7)-(9) imply

(10) $Y_{i+1}$ *is a continuum and* $r_{k+1}$ *is monotone.*

If $\pi_{k+1}(x) = g_k r_k(y)$ and $y\in X'_k$, then $\pi'_{k+1}(x, y) = y$ and

$$g_k^{-1} \pi_{k+1} g_{k+1}(x, y) = g_k^{-1} \pi_{k+1}(x) = g_k^{-1} g_k r_k(y) = g_k^{-1} g_k(y) = y.$$

Hence

(11) $\pi'_{k+1} = g_k^{-1} \pi_{k+1} g_{k+1}$.

In particular, if $(x, y)\in Y_{i+1}$, then $\pi_{k+1}(x) = g_k r_k(y)$; thus $r_k \alpha_{k+1}(x, y) = r_k(y)$ and

$$\pi'_{k+1} r_{k+1}(x, y) = g_k^{-1} \pi_{k+1} g_{k+1} g_{k+1}^{-1} \beta(x, y) = g_k^{-1} \pi_{k+1}(x) = g_k^{-1} g_k r_k(y) = r_k(y).$$

Thereby,

(12) $r_k \alpha_{k+1} = \pi'_{k+1} r_{k+1}$.

Now, let $y\in Y_k$ and $(x', y')\in X'_{k+1}$ be such that $r_k(y) = \pi'_{k+1}(x', y')$. Since $\pi'_{k+1}(x', y') = r_k(y') = y'$, we obtain $r_k(y) = y'$. Take $t = (x', y)$. Then equalities $g_k r_k(y) = g_k(y') = g_k r_k(y') = \pi_{k+1}(x')$ give $t\in Y_{k+1}$. Since $r_{k+1}(x', y) = g_{k+1}^{-1}(x') = r_{k+1}(x', y')$ and $\alpha_{k+1}(x', y) = y$, we infer that condition (4) holds for $i = k+1$ by (11) and (12).

Let $(x_n, y_n)\to(x, y)$, $(x_n, y_n)$, $(x, y)\in X'_{k+1}$ and $(x', y')\in r_{k+1}^{-1}(x, y)$. Then $\pi'_{k+1}(x_n, y_n) = y_n\in X'_k$, $\pi'_{k+1}(x, y) = y\in X'_k$, $y_n\to y$, $\alpha_{k+1}(x', y') = y'\in Y_k$, and $r_k(y') = y$. Since $r_k$ is open, there are $y'_n\in r_k^{-1}(y_n)$ such that $y'_n\to y'$. Since

$$x = g_{k+1} g_{k+1}^{-1}(x) = g_{k+1} g_{k+1}^{-1} \beta(x, y) = g_{k+1} r_{k+1}(x, y) = g_{k+1} r_{k+1}(x', y')$$
$$= g_{k+1} g_{k+1}^{-1} \beta(x', y') = g_{k+1} g_{k+1}^{-1}(x') = x',$$

we obtain $(x_n, y'_n)\to(x', y')$. Moreover, $(x_n, y'_n)\in Y_{k+1}$ because $g_k r_k(y'_n) = g_k(y_n) = g_k r_k(y_n) = \pi_{k+1}(x_n)$; and $(x_n, y'_n)\in r_{k+1}^{-1}(x_n, y_n)$ because $g_{k+1} r_{k+1}(x_n, y'_n) = x_n = g_{k+1} r_{k+1}(x_n, y_n)$. This completes the proof that $r_{k+1}$ is open.

It remains to prove condition (3) for $i = k+1$. Observe first that

(13) $\alpha_{k+1}^{-1}(Y_k\backslash X'_k) = Y_{k+1}\backslash X'_{k+1}$.

Indeed, if $(x, y)\in Y_{k+1}$ and $\alpha_{k+1}(x, y)\in X'_k$, then $\alpha_{k+1}(x, y) = y\in X'_k$, i.e., $(x, y)\in X'_{k+1}$ by the definition of $X'_{k+1}$.

Now, let $y\in Y_{k+1}\backslash X'_{k+1}$. Then $\alpha_{k+1}(y)\in Y_k\backslash X'_k$. Therefore, there are closed-open sets $V_n$ in $Y_k\backslash X'_{k+1}$ such that $\bigcap_{n=1}^{\infty} r_k(V_n) = r_k(\alpha_{k+1}(y))$, $V_{n+1}\subset V_n$, and $\alpha_{k+1}(y) \in V_n$ for $n = 1, 2, \ldots$ by (3) for $i = k$. Sets $W_n = \alpha_{k+1}^{-1}(V_n)$ are closed-open in $Y_{k+1} \setminus X'_{k+1}$ by (13).

Since $X'_{k+1}$ is closed in $Y_k$, sets $V_n$ are open in $Y_k$. We infer that sets $\pi_{k+1}^{-1} g_k r_k(V_n)$ are open in $X_{k+1}$. Furthermore,

$$g_{k+1} r_{k+1}(y) \in \pi_{k+1}^{-1} g_k r_k \alpha_{k+1}(y) = \bigcap_{n=1}^{\infty} \pi_{k+1}^{-1} g_k r_k(V_n).$$

Since the set $\pi_{k+1}^{-1} g_k r_k \alpha_{k+1}(y)$ is finite, for each $n = 1, 2, \ldots$ we can find the decomposition of the set $\pi_{k+1}^{-1} g_k r_k(V_n)$ into two disjoint open sets $G_n$ and $H_n$ such that

$$g_{k+1} r_{k+1}(y) = \bigcap_{n=1}^{\infty} G_n.$$

Take $V'_n = r_{k+1}^{-1} g_{k+1}^{-1}(G_n) \cap H_n$ for $n = 1, 2, \ldots$ Every set $V'_n$ is open. It is closed because its complement in $W_n$ is open, being equal to the set $r_{k+1}^{-1} g_{k+1}^{-1}(H_n) \cap W_n$.

Let

$$\alpha_{i,j} = \alpha_i \alpha_{i+1} \ldots \alpha_j, \qquad \pi'_{i,j} = \pi'_i \pi_{i+1} \ldots \pi'_j \quad \text{for } i < j,$$

$$Y = \operatorname{Inv}\lim\{Y_i, \alpha_{i,j}\}, \qquad X' = \operatorname{Inv}\lim\{X'_i, \pi'_{i,j}\},$$

let $r$ be a mapping from $Y$ onto $X'$ induced by $\{r_i\}$, and $g$ a mapping from $X'$ onto $X$ induced by $\{g_i\}$. Of course, $g$ is a homeomorphism from $X'$ onto $X$. It follows from Theorem 4 in [20], p. 61 (cf. [4]), that $r$ is an open monotone retraction from $Y$ onto $X'$.

Let $\delta_i$ denote the projection from $Y$ onto $Y_i$ and $y \in Y \setminus X'$. Then $\delta_i(y) \in Y_i \setminus X'_i$ for each $i = 1, 2, \ldots$ It follows from (3) that the quasi-component $F$ of the point $y$ in $Y \setminus X'$ is contained in $\delta_i^{-1}(r_i(\delta_i(y)))$ for each $i = 1, 2, \ldots$ Therefore, $r(F)$ is a one-point set, i.e., (iii) holds.

If $F$ is a continuum in $Y \setminus X'$, then $r(F)$ is degenerate. From (5) we conclude that $\delta_1 | r^{-1}(t)$ is a homeomorphism from $r^{-1}(t)$ onto $r_1^{-1}(r_1 \delta_1(t))$. Therefore, the composition $h = \mu \circ \delta_1$ has all required properties, and the proof of Theorem 5 is complete.

From Theorem 5 we obtain the more general result:

COROLLARY 1. *Let $X$ and $N$ be arbitrary continua and $t_0 \in N$. Then there are a continuum $Y$ containing $X$, a monotone open retraction $r$ from $Y$ onto $X$, and a continuous mapping $h$ from $Y$ onto $N$ such that*

(i) *$r^{-1}(x)$ is non-degenerate for each $x \in X$,*

(ii) *$h^{-1}(t_0) = X$,*

(iii) *if $F$ is a quasi-component of $Y \setminus X$, then $F \subset r^{-1}(x)$ for some $x \in X$,*

(iv) *if $F \subset Y \setminus X$ is a continuum, then $h|F$ is a homeomorphism.*

Proof. It is known (see [22]) that if $\mathcal{U}$ is the Menger universal curve and $Q$ is the Hilbert cube, then there is an open and monotone mapping $\alpha$

from $\mathscr{U}$ onto $Q$. We can assume that $X \subset Q$. It follows from Theorem 5 that there are a continuum $M$ containing $\mathscr{U}$, a monotone open retraction $\omega$ from $M$ onto $\mathscr{U}$, and a continuous mapping $\mu$ from $M$ onto $N$ such that $\omega^{-1}(\mu)$ is non-degenerate for each $\mu \in \mathscr{U}$, $\mu^{-1}(t_0) = \mathscr{U}$, if $F$ is a quasi-component of $M \backslash \mathscr{U}$, then $F \subset \omega^{-1}(\mu)$ for some $\mu \in \mathscr{U}$, and if $F \subset M \backslash \mathscr{U}$ is a continuum, then $\mu | F$ is a homeomorphism.

Let $M' = \omega^{-1} \alpha^{-1}(X)$ and define an equivalence relation $\sim$ on $M'$ by $m \sim m' \Leftrightarrow m = m'$ or $m, m' \in \mathscr{U}$ and $\alpha(m) = \alpha(m')$. Put $Y = M'/\sim$ and let $\varphi$: $M' \to Y$ be the quotient map. Define $X' = \varphi(\mathscr{U} \cap M')$, $r(y) = \varphi\omega\varphi^{-1}(y)$ and $h(y) = \mu\varphi^{-1}(y)$ for $y \in Y$ and $g(y) = \alpha\omega\varphi^{-1}(y)$ for $y \in X'$.

One can easily check that $g$ is a homeomorphism from $X'$ onto $X$ and that $Y$, $r$, and $h$ satisfy all required conditions. The openness of $r$ is the most doubtful, thus we check only it. It suffices to show that if $x_n \to x$, $x_n$, $x \in X'$, and $y \in r^{-1}(x)$, then there are $y_n \in r^{-1}(x_n)$ such that $y_n \to y$. Let $m \in \varphi^{-1}(y)$ and $a \in \varphi^{-1}(x) \cap \mathscr{U}$. Since $g(x) = \alpha\omega\varphi^{-1}(x) = \alpha\omega(a) = \alpha(a)$ and $\varphi(a) = x = r(y) = \varphi\omega\varphi^{-1}(y) = \varphi\omega(m)$, we obtain $g(x) = \alpha(a) = \alpha\omega(m)$. Thus, since $g(x_n) \to g(x) = \alpha\omega(m)$ and $\alpha$ is open, there are $\mu_n \in \mathscr{U}$ such that $\alpha(\mu_n) = g(x_n)$ and $\mu_n \to \omega(m)$. Since $\omega$ is open, there are $z_n \in \omega^{-1}(\mu_n)$ such that $z_n \to m$. Then $\varphi(z_n) \to \varphi(m) = y$. But

$$\begin{aligned} gr\varphi(z_n) &= \alpha\omega\varphi^{-1}\varphi\omega\varphi^{-1}\varphi(z_n) = \alpha\omega\varphi^{-1}\varphi\omega(z_n) \\ &= \alpha\omega\varphi^{-1}\varphi(\mu_n) = \alpha\omega(\mu_n) = \alpha(\mu_n) = g(x_n). \end{aligned}$$

Since $g$ is a homeomorphism, we have $r\varphi(z_n) = x_n$, i.e., $\varphi(z_n) \in r^{-1}(x_n)$, which completes the proof.

Recall that if $X$ is a continuum, then a decomposition $\mathscr{D}$ of $X$ is said to be *admissible* if $\mathscr{D}$ is upper semi-continuous and monotone and for every irreducible continuum $I$ in $X$ every layer $T_t$, $0 \leqslant t \leqslant 1$, of $I$ is contained in some element of $\mathscr{D}$. If $\mathscr{U}$ is a hereditarily arcwise connected continuum, then $\mathscr{U}$ is a curve. Therefore, taking a pseudo-arc as the continuum $N$ in Theorem 5 we obtain the following solution of problems of Krasinkiewicz and Charatonik (see [2], Problems 2 and 3, p. 148).

COROLLARY 2. *If $Y$ is a hereditarily arcwise connected continuum, then there exists a continuum $X$ every stratum of which is a non-trivial stratum of continuity and such that $Y$ is the decomposition space of the canonical admissible decomposition of $X$.*

**3. Strongly homotopically stable mappings.** A mapping $f$: $X \to Y$ of spaces is said to be *strongly homotopically stable* if $f$ homotopic to $g$: $X \to Y$ implies $f = g$.

Recall that a mapping $f$ from a space $X$ onto a space $Y$ is *locally confluent* if for each point $y$ of $Y$ there is an open set $\mathscr{U}$ containing $y$ such that $f | f^{-1}((\mathrm{Cl}\,\mathscr{U}))$ is confluent.

THEOREM 6. *Suppose $f$ is a locally confluent and light mapping of a compactum $X$ onto a compactum $Y$. Then there are arbitrarily small closed neighbourhoods $\mathcal{U}$ of $x$ in $X$ such that $f|\mathcal{U}$ is confluent.*

Proof. Since $f$ is light, there is a finite open (in $X$) covering $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $f^{-1}f(x)$ such that the union of the boundaries of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is disjoint with $f^{-1}(f(x))$. Then

$$B = \bigcup_{k=1}^{n} f(\operatorname{bd} \mathcal{U}_k) \subset Y \setminus \{f(x)\}.$$

Therefore, we can find a closed neighbourhood $V$ of $f(x)$ which is disjoint with $B$ and such that $f|f^{-1}(V)$ is confluent. Consider the sets $A_i = \mathcal{U}_i \cap f^{-1}(V)$. Then $f(A_i) = f(\mathcal{U}_i) \cap V$. Since the components of $f^{-1}(K)$ for each continuum $K \subset V$ are disjoint with the boundaries of $\mathcal{U}_1, \ldots, \mathcal{U}_k$, each such component must be contained in $A_i$ or disjoint with it. Therefore, $f|A_i$ is confluent for $i = 1, \ldots, n$, which completes the proof.

The following corollary answers in the affirmative Problem 1010 of [12].

COROLLARY 3. *If $f: X \to Y$ is a light open mapping of compacta, then for each $x_0 \in X$ there exist arbitrarily small closed neighbourhoods $U$ of $x_0$ such that $f(x_0) \in \operatorname{Int} f(U)$ and $f|U$ is confluent.*

The next corollary gives a positive solution to Problem 16 of UHPB.

COROLLARY 4. *Suppose $f$ is a locally confluent and light mapping of a compactum $X$ onto a locally connected compactum $Y$. Then for each $x_0 \in X$ there exist arbitrarily small closed neighbourhoods $U$ of $x_0$ such that $f(x_0) \in \operatorname{Int} f(U)$ and $f|\mathcal{U}$ is confluent.*

Proof. This follows from Theorem 6 and the observation that the closed neighbourhood $V$ in, the proof of Theorem 6 can be taken to be connected.

By a *curve* is meant a one-dimensional continuum. A curve is said to be *locally acyclic* at a point $x_0 \in X$ provided there exists a subset $A$ of $X$ such that $x_0 \in \operatorname{Int} A$ and each mapping of $A$ into the circle is homotopic to a constant mapping.

The following is an extension of a result of Lelek ([12], 3.3) to non-locally connected spaces. It is an immediate consequence of Corollary 3 and [12], 2.2.

COROLLARY 5. *If $f: X \to Y$ is a light open mapping of a compactum $X$ onto a curve $Y$ such that $Y$ is not locally acyclic at any point, then $f$ is strongly homotopically stable.*

The next example shows that the condition in Corollary 5 that $f$ be light is necessary. It also answers in the negative Problems 15 and 18 in UHPB.

Example 2. Let $Z$ be a continuum constructed as in Theorem 5 so that $Z$ admits a monotone open retraction $\varphi: Z \to S$, where $S$ is a Sierpiński universal curve, $\varphi^{-1}(x)$ is a smooth fan for each $x \in S$, the set of end-points of $Z$ is zero-dimensional, and each component of $Z \backslash S$ is a homeomorph of $(0, 1]$ whose closure meets $S$ in exactly one point (such $Z$ we obtain taking $N = [0, 1]$ and $t_0 = 0$ in Theorem 5). We indicate a proof that $\varphi$ is *not* strongly homotopically stable.

Let $a$ and $b$ be end-points of the closure of some component of $Z \backslash S$ with $b \in S$ and let $f$ be an arbitrary mapping from $S \cup \{a\}$ into $S$ such that $f|S$ is the identity and $f(a) \neq b$. From Theorem 1 in [9], p. 347, we conclude that there is a mapping $f^*\colon Z \to S$ which is an extension of $f$. Then $f^* \neq \varphi$ and $f^*$ is homotopic to $\varphi$.

We complete this section by giving an example to show that the condition in Corollary 4 that $Y$ be locally connected is necessary. The example also answers in the negative Problem 36 in UHPB.

Example 3. Let $(x, y)$ denote a point of the Euclidean plane having $x$ and $y$ as its rectangular coordinates and let $P$ denote the standard $(\sin \pi/x)$-curve, i.e.,

$$P = \{(x, \sin \pi/x)\colon\ 0 < x \leqslant 1\}.$$

Consider the set

$$X = \{(x, y)\colon\ (x, y-1) \in P \text{ or } (-x, 2(y+3/2)) \in P\} \cup \{(0, y)\colon\ -2 \leqslant y \leqslant 2\}$$

and a mapping $f(x, y) = (x, |y|)$. Then $f$ is locally confluent and weakly confluent. No small neighbourhood of $(0, 2)$ in $X$ is mapped onto a neighbourhood of $(0, 2)$ in $f(X)$ and there is no subcontinuum $L$ of $X$ such that $f(L) = f(X)$ and $f|L$ is confluent.

**4. Continuous decompositions of plane curves.** Let

$$P = \mathrm{Cl}\{(x, \sin 1/x)\colon\ 0 < x \leqslant 1\} \cup \mathrm{Cl}\{(x, \sin 1/(2-x))\colon\ 1 \leqslant x < 2\}.$$

and for $\varepsilon > 0$

$$Q^\varepsilon = \mathrm{Cl}\{(x, y)\colon\ \sin 1/x - \varepsilon x \leqslant y \leqslant \sin 1/x,\ 0 < x \leqslant 1\} \cup$$
$$\cup\, \mathrm{Cl}\{(x, y)\colon\ \sin 1/(2-x) - (2-x)\varepsilon \leqslant y \leqslant \sin 1/(2-x),\ 1 \leqslant x < 2\}.$$

We can call $\{0\} \times [-1, 1]$ and $\{2\} \times [-1, 1]$ the *end slices* of $Q^\varepsilon$. It is easy to see that $Q$ has a continuous decomposition onto an arc so that each fiber of the decomposition is an arc of diameter at least $2 - 2\varepsilon$.

Krasinkiewicz [7], Theorem 3.4, proved that if $X$ is a continuum which is obtainable as the continuous decomposition of a plane continuum where each fiber of the decomposition is a decomposable non-degenerate continuum, then each cyclic element $K$ of $X$ is a plane completely regular continuum (i.e., each non-degenerate continuum in $K$ has non-void interior in $K$). Krasinkiewicz asked in [7] (Problem 1) and in UHPB (Problem 119) whether the converse to this theorem is true. In the next theorem we shall give a partial affirmative solution to this problem.

We remark that the condition that the fibers be decomposable in Krasinkiewicz's theorem is necessary since every plane continuum can be obtained as the continuous decomposition of a plane continuum with all fibers being pseudo-arcs [14].

THEOREM 7. *If $X$ is a completely regular plane continuum, then there exist a plane continuum $Y$ and a monotone open mapping $f\colon Y \to X$ such that $f^{-1}(x)$ is an arc for each $x \in X$.*

Proof. By [5], Lemma 2, we have $X = F \cup A_1 \cup A_2 \cup \ldots$, where $F$ is a Cantor set, each $A_i$ is an arc with end-points $a_i$, $b_i$ such that $F \cap A_i = \{a_i, b_i\}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $A_i \setminus \{a_i, b_i\}$ is open in $X$ for each $i$, and the sequence $(A_i)$ is a null sequence.

We may suppose the space $X$ is embedded in the plane $R^2$ in such a way that $F = C$ is a Cantor's ternary set in the interval $[0, 1] \times \{0\}$. Let $\pi_1\colon R^2 \to R \times \{0\}$ and $\pi_2\colon R^2 \to \{0\} \times R$ be the natural projections onto the $x$-axis and $y$-axis, respectively.

Notice that for each component $U$ of $[0, 1] \times \{0\} \setminus C$ there exist at most finitely many integers $i$ such that $\pi_1(A_i) \supset U$. We may suppose (since $(A_i)$ is a null sequence) that for each $i$ there exist sequences $(a_{i,j})$ and $(b_{i,j})$ in $A_i$ convering to $a_i$ and $b_i$, respectively, such that $\pi_2(a_{i,j})$, $\pi_2(b_{i,j}) > 0$ for odd $j$ and $\pi_2(a_{i,j})$, $\pi_2(b_{i,j}) < 0$ for even $j$.

We may suppose that if $(c_j)$ is a convergent sequence of distinct points in $A_i \cap [0, 1] \times \{0\}$, then $\lim c_j \in \{a_i, b_i\}$. We suppose that if $A$ is an arc contained in $A_i \setminus \{a_i, b_i\}$, then $A$ is rectifiable. Finally, we suppose that if $c \in A_i \cap [0, 1] \times \{0\}$ and $c \notin \{a_i, b_i\}$, then each neighbourhood of $c$ in $A_i$ meets both $\pi_2^{-1}(0, \infty)$ and $\pi_2^{-1}(-\infty, 0)$. Define $\theta\colon R^2 \to R^2$ by

$$\theta(x, y) = \begin{cases} (x, y) & \text{if } y \leqslant 0, \\ (x, 0) & \text{if } 0 \leqslant y \leqslant 1, \\ (x, y-1) & \text{if } 1 \leqslant y. \end{cases}$$

Let $Y^1 = \theta^{-1}(X)$. Then $\theta^{-1}(x, 0)$ is an arc for each $(x, 0) \in C$ and $\theta^{-1}(A_i)$ is homeomorphic to $P$ for each integer $i$. For each $i$ there exist continua $Q_i$ in $R^2$ homeomorphic to $Q^1$ such that $\theta^{-1}(a_i)$ and $\theta^{-1}(b_i)$ are the end slices of $Q_i$, $\theta^{-1}(A_i)$ is contained in the boundary of $Q_i$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$, $Q_i \cap Y^1 = \theta^{-1}(a_i) \cup \theta^{-1}(b_i)$, $Q_i$ admits a $(1/i)$-retraction onto $A_i$, and $Q_i$ admits a continuous decomposition whose fibers are arcs of diameter greater than or equal to $1-2/(i+2)$ onto an arc. Then $Y = Y^1 \cup Q_1 \cup Q_2 \cup \ldots$ is a plane continuum which admits a continuous monotone decomposition onto $X$ and such that each fiber is an arc.

**5. Further examples.** Remark that Example 2 in [16] solves negatively Problem 100 in PBUH (this problem concerns open' mappings on smooth dendroids).

Example 4. Let $B$ denote a Knaster's indecomposable continuum with only one end-point $b$. Joining two disjoint copies of $B$ at the point $b$ we obtain an arc-like continuum $R$ without end-points. It is known that there is a continuous mapping $f$ from a pseudo-arc $P$ onto $R$ (see [10]) and every such mapping is weakly confluent. There is no point $p$ in $P$ such that if $K$ is a subcontinuum of $R$ containing $f(p)$ and $C$ is a component of $f^{-1}(K)$ containing $p$, then $f(C) = K$ (by the hereditary indecomposability of $P$). This solves Problems 55 and 56 in PBUH.

Example 5. It is known (see [3]) that there are a hereditarily decomposable chainable continuum $K \subset \{(x, y) \in R^2 \colon x \geqslant 0\}$ and a weakly confluent mapping $f$ from $K$ onto a simple triod $T = \{(x, 0) \in R^2 \colon 0 \leqslant x \leqslant 1\} \cup I$, where $I = \{(0, y) \colon -1 \leqslant y \leqslant 1\}$. Moreover, if $\varphi$ is a canonical monotone mapping from $K$ onto $[0, 1]$, then $\varphi^{-1}(t)$ are arcs, $\varphi^{-1}(0) = I$, $f(\varphi^{-1}(0)) = I$, and $f(\varphi^{-1}([0, t))) \neq T$ for $t < 1$.

Take $X = K \cup \{(x, \sin \pi/x) \colon -1 \leqslant x < 0\}$ and a mapping $f^*$ from $X$ onto $f^*(X)$ such that $f^*(x) = f(x)$ for $x \in K$ and $\{x\} = f^{-1} f^*(x)$ for $x \in X \setminus K$. The set $X$ is a chainable continuum, $f^*(X)$ is a non-chainable continuum, $f^*$ is weakly confluent, and there exist no subcontinuum $X'$ of $X$ mapped onto $f^*(X)$ under $f^*$ (each such $X'$ is equal to $X$ in our example) such that $f^* | X'$ is not weakly confluent. In this way Problem 58 in PBUH has a negative answer.

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