COLLOQUIUM MATHEMATICUM

VOL. XXIX 1974 FASC. 2

A FACTORIZATION LEMMA AND ITS APPLICATION TO REALIZATION OF MAPPINGS AS INVERSE LIMITS

BY

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The aim of this paper* is to prove a factorization lemma for mappings (1) of compact metric spaces into polyhedra and to apply it in proving that each mapping between compact metric spaces can be realized by a sequence of mappings between some polyhedral expansions of given spaces. Such a result was stated by Kaul in [2], but in our paper we consider the case where the given mapping is onto and we prove that mappings of the realizing sequence can also be constructed to be onto; e.g., the known theorem of Freudenthal [1] on polyhedral expansions with onto mappings is contained in our theorem.

1. Introduction. A mapping f: $X \to Y$ is said to be *realized* by a sequence of mappings f_n: $X_n \to Y_n$, $n = 1, 2, \ldots$ (see Kaul [2]), if there exist inverse sequences

$$X_1 \xleftarrow{\pi_1^2} X_2 \xleftarrow{\pi_2^3} \ldots \tag{1}$$

and

$$Y_1 \xleftarrow{\sigma_1^2} Y_2 \xleftarrow{\sigma_2^3} \ldots \tag{2}$$

whose limits are X and Y, respectively, and if for each $x \in X$ and $n = 1, 2, \ldots$ there is

$$\lim_{r\to\infty} \sigma_n^r\big(f_r(\pi_r(x))\big) = \sigma_n f\big((x)\big). \tag{3}$$

The question (see Mardešić [3], Remark 2, p. 247), which is still open, is whether any mapping f can be realized as a mapping induced by a sequence $f_1, f_2, \ldots$ (*induced* means that lim in (3) can be dropped).

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* This paper is a part of master's thesis written under the guidance of Prof. J. Mioduszewski.

(1) Throughout this paper all mappings are assumed to be continuous.

The following two notions are taken from [2]. The second notion is modified here by deleting one of the conditions.

Let α and β be barycentric mappings of spaces X and Y into polyhedra (P, T) and (P', T'), respectively, and let g be a mapping of X into Y. A mapping g' of P into P' is said to be a *carrier mapping with respect to* g if, for every $x \in X$,

$$g'\big(\mathrm{car}_{(P,T)}\,\alpha(x)\big) \subset \mathrm{car}_{(P',T')}\,\beta\big(g(x)\big),$$

where $\mathrm{car}_{(P,T)}p$ denotes the carrier of a point p in a polyhedron (P, T), i.e. a simplex $s \in T$ such that p lies in the geometrical interior of s.

Let $U_1, U_2, \ldots$ be a sequence of finite open coverings of a space X. An inverse sequence $\{N(U_n);\ \pi_{m,n}\}$, where $N(U_n)$ denotes the nerve of U_n, is called an *auxiliary inverse sequence associated with* X if $\pi_{m,n}$ and $N(U_n)$ satisfy the following conditions:

(a) $\mathrm{mesh}\, U_n \to 0$ if $n \to \infty$;

(b) $\pi_{n+1,n}\colon N(U_{n+1}) \to N(U_n)$ is a carrier mapping with respect to the identity on X;

(c) if σ_m is an arbitrary simplex from $N(U_m)$, then, for every natural n, $\mathrm{diam}\, \pi_{m,n}(\sigma_m) \to 0$ if $m \to \infty$.

In this note the following lemma is proved:

FACTORIZATION LEMMA. *Let X be a compact metric space. If there are given polyhedra (P, T) and (P', T'), mappings*

$$f\colon X \xrightarrow{\text{onto}} P \quad \text{and} \quad f'\colon X \xrightarrow{\text{onto}} P',$$

and a positive number ε, then there exist a polyhedron (Q, S) such that

$$\dim Q \leqslant \max(\dim X, \dim P, \dim P'),$$

an ε-mapping $g\colon X \xrightarrow{\text{onto}} Q$, and simplicial mappings

$$\pi\colon (Q, S) \xrightarrow{\text{onto}} (P, T) \quad \text{and} \quad \pi'\colon (Q, S) \xrightarrow{\text{onto}} (P', T'),$$

which are carriers with respect to the identity on X.

Remark 1. From the fact that π is a carrier mapping with respect to the identity on X it follows that, for each $x \in X$, $\pi\big(g(x)\big)$ and $f(x)$ are in the same simplex of T, which implies that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & P \\ Q \arrow[ur, "\pi"'] & \end{tikzcd}$$

is $(\mathrm{mesh}\, T)$-commutative (of course, the same is also true for π' with respect to f' and P').

Remark 2. It follows immediately from the proof of Factorization Lemma that if we do not require mappings π and π' to be simplicial, then the polyhedron Q can be constructed to have the dimension not greater than $\max(\dim X, 1)$.

Analogous factorization lemmas are given in Freudenthal [1] (see Hilfssatz 13, p. 191), and in Mardešić and Segal [4] (see Lemma 4): in [1] for f being irreducible in a sense defined there, and in [4] for X being a continuum (with a somewhat stronger thesis, so that the result is not comparable with our lemma).

As an application of our Factorization Lemma we get

THEOREM. *Let X and Y be compact metric spaces and let $f\colon X \to Y$ be a mapping. Then there exist*

1° *auxiliary inverse sequences $\{N(U_n); \pi_{m,n}\}$ and $\{N(V_n); \psi_{m,n}\}$ associated with X and Y, respectively, such that $\dim N(U_n) \leqslant \dim X$, $\dim N(V_n) \leqslant \dim Y$, $\pi_{m,n}$ and $\psi_{m,n}$ are simplicial mappings of $N(U_m)$ and $N(V_m)$, respectively, onto a certain subdivision of $N(U_n)$ and $N(V)_n$, respectively, and*

2° *a sequence of simplicial mappings $f_n\colon N(U_n) \to N(V_n)$ which realizes f.*

In addition, if f is onto and (a) *if we replace the condition $\dim N(U_n) \leqslant \dim X$ by the condition $\dim N(U_n) \leqslant \max(\dim X, \dim Y, 1)$, or* (b) *if we do not require mappings f_n to be simplicial, then mappings f_n can be constructed to be onto.*

The theorem is a stronger version of Kaul's Theorem A [2]. In particular, we assert that the mappings $\pi_{m,n}$ and $\psi_{m,n}$ are onto, and, if f is onto, so are the mappings f_n.

2. Proof of Factorization Lemma. Let

$$\max(\dim X, \dim P, \dim P') > 0$$

(otherwise the proof is trivial).

In view of the symmetry of hypotheses concerning f, P, T and f', P', T', if whatever will be defined (will be true) for one of these triples, for given ε, g and Q, then it will be assumed to be defined (to be true) also for the other; if the symbol of the notion defined for f, P, T is a, then the corresponding symbol for f', P', T' will be denoted by a', and *vice versa*, so a'' will be meant as a.

Let γ be a Lebesgue number for the covering $\{f^{-1}(\mathrm{St}_T b)\colon b \text{ is a vertex of } T\}$ of X and let $\varepsilon_2 = \frac{1}{2}\min(\gamma, \gamma', \varepsilon)$.

Let $s_1, \ldots, s_m$ be all maximum simplices of T such that the dimension of each of them is greater than 0, let $a_1, \ldots, a_m$ be a certain choice of vertices from $s_1, \ldots, s_m$ and let ε_1 and U_i^j ($j \leqslant 6$, $i \leqslant m$) be a positive number and open sets, respectively, such that $\varepsilon_1 \leqslant \varepsilon_2$,

$\mathrm{Cl}\, U_i^1 \subset U_i^2$, $\mathrm{Cl}(U_i^2 \cup U_i^3) \subset U_i^4 \subset \mathrm{Cl}\, U_i^4 \subset U_i^5 \subset \mathrm{Cl}\, U_i^5 \subset U_i^6 \subset \mathrm{Cl}\, U_i^6 \subset f^{-1}(\mathrm{int}\, s_i)$, where $\mathrm{int}\, s_i$ is the geometrical interior of s_i,

$$\mathrm{Cl}\, U_i^2 \cap \mathrm{Cl}\, U_i^3 = \emptyset,$$

$$\mathrm{card}\, U_i^1 = \mathrm{card}\, U_i^3 = \text{continuum}, \quad \mathrm{Cl}\, U_i^5 \neq U_i^6, \quad \mathrm{diam}\ U_i^6 < \varepsilon_1,$$

$$\mathrm{dist}\big(U_i^6, \mathrm{Fr}\, f^{-1}(\mathrm{int}\, s_i)\big) > 2\varepsilon_1$$

and

$$\mathrm{Cl}\, U_i^6 \cap \mathrm{Cl}[(U_1^6)' \cup (U_2^6)' \cup \ldots \cup (U_{m'}^6)'] = \emptyset.$$

Let

$$Y = X - [U_1^5 \cup \ldots \cup U_m^5 \cup (U_1^5)' \cup \ldots \cup (U_{m'}^5)'].$$

Let $\{V_1, \ldots, V_p, V_1', \ldots, V_{p'}'\}$ be a covering of Y such that all V_i and $V_{i'}'$ are open subsets of

$$X - \mathrm{Cl}[U_1^4 \cup \ldots \cup U_m^4 \cup (U_1^4)' \cup \ldots \cup (U_{m'}^4)']$$

such that their diameters are less than ε_1, each point of X belongs to at most $n+1$ of them, where n is a certain integer not greater than $\dim X$, and $\emptyset \neq V_i \subset U_i^6$ for $i = 1, 2, \ldots, m$.

Let f_i^1, f_i^2 ($i \leqslant m$) and g_j ($j \leqslant p$) map X into $I = [0,1]$ as follows:

$$f_i^1(\mathrm{Cl}\, U_i^1) = I, \quad f_i^1(\mathrm{Cl}\, U_i^5 - U_i^2) = \{1\}, \quad f_i^1(X - U_i^6) = \{0\},$$

$$f_i^2(\mathrm{Cl}\, U_i^3) = I, \quad f_i^2(\mathrm{Cl}\, U_i^2) = \{1\}, \quad f_i^2(X - U_i^4) = \{0\},$$

$$g_j(X - V_j) \subset \{0\}, \quad \sum_{j=1}^{p} g_j(x) + \sum_{j=1}^{p'} g_j'(x) \neq 0 \quad \text{for every } x \in Y.$$

Let N denote a realization of the nerve of the family $\{V_1, \ldots, V_p, V_1', \ldots, V_{p'}'\}$ lying in a $(2n+1)$-plane $\sigma \subset R^{2n+2}$. Let $s(V_i)$ denote the 1-simplex lying in R^{2n+2}, which is perpendicular to σ and intersects σ at the point V_i, and let all opposite vertices $\dot{V}_i$ of $s(V_i)$ and $\dot{V}_{i'}'$ of $s(V_{i'}')$ lie on the $(2n+1)$-plane σ_0 which is parallel to σ.

Let

$$N_1 = s(V_1) \cup \ldots \cup s(V_m) \cup s(V_1') \cup \ldots \cup s(V_{m'}') \cup N.$$

Now we shall define a mapping h of X into N_1.

1° If $x \in \mathrm{Cl}\big(X - (U_1^5 \cup \ldots \cup U_m^5 \cup (U_1^5)' \cup \ldots \cup (U_{m'}^5)')\big)$, then $h(x) \in N$ and a V_i-coordinate $h_{V_i}(x)$ of $h(x)$ is given by the formula

$$h_{V_i}(x) = \begin{cases} \dfrac{g_i(x) + f_i^1(x)}{A} & \text{for } i \leqslant m, \\[2ex] \dfrac{g_i(x)}{A} & \text{for } m < i \leqslant p, \end{cases}$$

where

$$A = \sum_{j=1}^{p} g_j(x) + \sum_{j=1}^{p'} g'_j(x) + \sum_{j=1}^{p} f^1_j(x) + \sum_{j=1}^{p'} (f^1_j)'(x)$$

($h_{V'_{i'}}(x)$ analogously).

2° If $x \in \mathrm{Cl}\, U^5_i$ $(i \leqslant m)$, then

$$h(x) = \left(\frac{f^1_i(x)}{f^1_i(x)+f^2_i(x)}, \frac{f^2_i(x)}{f^1_i(x)+f^2_i(x)}\right) \in s(V_i)$$

(analogously if $x \in \mathrm{Cl}((U^5_{i'})')$).

Now we shall show that h is an ε_1-mapping.

Let $h(x) = h(y)$.

Case (a). There exists an i such that x and y belong to U^6_i or to $(U^6_{i'})'$. Then $\varrho(x, y) \leqslant \mathrm{diam}\, U^6_i < \varepsilon_1$ (respectively, $\varrho(x, y) \leqslant \mathrm{diam}(U^6_{i'})' < \varepsilon_1$).

Case (b). The points x and y belong neither to the same U^6_i nor to the same $(U^6_{i'})'$.

If there exists an i such that $f^1_i(x) \neq 0$, then $x \in U^6_i, f^1_i(y) + g_i(y) \neq 0$ and, because $f^1_i(y) = 0$ (which follows from $y \notin U^6_i$), we get consequently $g_i(y) \neq 0$. Then $y \in V_i$, and x and y belong to U^6_i, because $V_i \subset U^6_i$ for $i \leqslant m$. A contradiction. Thus

$$f^1_i(x) = f^1_i(y) = (f^1_{i'})'(x) = (f^1_{i'})'(y) = 0 \quad \text{for } i \leqslant m \text{ and } i' \leqslant m'.$$

Consequently, $g_j(x) = g_j(y)$ and $g'_{j'}(x) = g'_{j'}(y)$ for every $j \leqslant p$ and $j' \leqslant p'$, and there exists a j_0 (or j'_0) such that $g_{j_0}(x) \neq 0$. But $g_{j_0}(X - V_{j_0}) \subset \{0\}$. Thus $x, y \in V_{j_0}$ and $\mathrm{diam}\, V_{j_0} < \varepsilon_1$. This completes the proof in case (b).

Since h is an ε_1-mapping, there exists a $\delta > 0$ such that

(i) if $A \subset N_1$ and $\mathrm{diam} A < \delta$, then $\mathrm{diam}\, h^{-1}(A) < 2\varepsilon_1$.

Let S_1 be a triangulation of N_1 such that, for every vertex b of S_1,

$$\mathrm{diam}\, \mathrm{St}_{S_1}(\mathrm{Cl}\, \mathrm{St}_{S_1} b) < \delta.$$

Let S_2 be a triangulation of N_1 which agrees with S_1 on N and which has only three vertices on $s(V_i)$: the vertex V_i, the first (if we count from the vertex V_i) vertex b_i of S_1 which is different from V_i and $\dot{V}_i$ (if such a vertex does not exist, then let b_i be the centre of $s(V_i)$) and the vertex $\dot{V}_i$. Now we improve h in order to get a $2\varepsilon_1$-mapping h_1 of X onto a certain subpolyhedron (N_2, S_3) of (N_1, S_2). This can be made in a standard way by applying successively sweeping of maximum simplices onto their boundaries if there exists an interior point of such a simplex not in the image of X. So we get

$$h_1\colon X \xrightarrow{\text{onto}} N_2.$$

We have

(ii) for every vertex b of S_3, $\operatorname{diam} \operatorname{St}_{S_3} b < \delta$;

(iii) $\operatorname{diam} h_1^{-1}\big(s(V_i) - \{V_i\}\big) < \varepsilon_1 < \varepsilon$.

Note that (ii) and (iii) imply

(iv) $h_1^{-1}(\operatorname{St}_{S_3} V_i) \subset f^{-1}(\operatorname{int} s_i)$.

Let $\tilde{s}_j$ denote the image of s_j under a certain linear embedding of s_j into σ_0 such that $\tilde{s}_i$ and $\tilde{s}_j$ do not intersect for $i \neq j$, and $\dot{V}_j$ corresponds to a_{i_j} under this embedding ($j \leqslant m$).

Let

$$Q = N_2 \cup \tilde{s}_1 \cup \ldots \cup \tilde{s}_m \cup \tilde{s}_1' \cup \ldots \cup \tilde{s}_{m'}'$$

and let S be a triangulation of Q which agrees with S_3 on N_2 and with the triangulation $(T|s_j)^{\sim}$ on $\tilde{s}_j$ ($j \leqslant m$).

We define a mapping

$$\varphi\colon N_2 \xrightarrow{\text{onto}} Q$$

in the following way:

1° $\varphi(x) = x$ for $x \in N_2 \cap \sigma$;

2° φ maps homeomorphically the arc $V_i b_i$ onto the arc $V_i \dot{V}_i$;

3° $\varphi(b_i \dot{V}_i) = \tilde{s}_i$ (and analogously on $V_{i'}' b_{i'}'$ and $b_{i'}' \dot{V}_{i'}'$).

Let $g = \varphi \circ h_1$ (in order to get the conclusion of Factorization Lemma in the form of Remark 2, it suffices to define g to be h_1; in this case the remaining part of the proof is as below with some obvious simplifications).

It follows from (i)-(iv) and from the definitions of ε_1 and δ that

(1) if b is a vertex of S which lies on σ, then

$$\operatorname{diam} g^{-1}(\operatorname{St}_S b) < \varepsilon \quad \text{ and } \quad g^{-1}(\operatorname{St}_S b) \subset f^{-1}(\operatorname{St}_T c)$$

for some vertex c from T;

(2) $\operatorname{diam} g^{-1}(\tilde{s}_i \cup V_i \dot{V}_i - \{V_i\}) < \varepsilon$;

(3) $g^{-1}(\operatorname{St}_S V_i) \subset f^{-1}(\operatorname{int} s_i)$.

Thus g is an ε-mapping of X onto Q.

Now we shall define a simplicial mapping π of (Q, S) onto (P, T).

Let b be a vertex of S. Then

1° if $b \in (\operatorname{St}_T a)^{\sim}$, $b = \tilde{c}$, where a and c are vertices of T, then $\pi(b) = c$;

2° $\pi(\dot{V}_j) = a_{i_j}$;

3° if $b \in N_2 \cap \sigma - \{V_1, \ldots, V_p, V_1', \ldots, V_{p'}'\}$, then $\pi(b) = c$, where c is chosen for b as in (1).

It is easy to verify that the just defined mapping of the set of vertices of Q into P has a simplicial extension and that π satisfies theses of our lemma (π' being defined according to the convention made at the beginning).

Thus the proof of the lemma is complete.

3. Realization of mappings as inverse limits.

LEMMA 1 (cf. [2], Theorem 1). *The inverse limit P of an auxiliary inverse sequence $\{N(U_n);\ \pi_{m,n}\}$ associated with a compact metric space X is homeomorphic to X.*

Proof. Let $\sigma_n(x)$ denote the carrier of x in $N(U_n)$. Let

$$\sigma(x) = \underleftarrow{\lim}\{\sigma_n(x);\ \pi_{m,n}|\sigma_m(x)\}.$$

Clearly, $\sigma(x) \neq \emptyset$. Let $p = (p_1, p_2, \ldots)$ and $q = (q_1, q_2, \ldots)$ be points of $\sigma(x)$. Then $p_n, q_n \in \pi_{m,n}(\sigma_m(x))$ for every $m > n$. It follows from condition (c) of the definition of an auxiliary inverse sequence that

$$\varrho(p_n, q_n) \leqslant \operatorname{diam} \pi_{m,n}(\sigma_m(x)) \to 0 \quad \text{ if } m \to \infty.$$

Hence $p_n = q_n$ for every n and $\sigma(x) = \{p\}$.

Now we define a mapping $a: P \to X$ by putting $a(p) = x$, where $\sigma(x) = \{p\}$.

It follows from Lemmas (2.1) and (2.4) of [2] that a is a homeomorphism.

LEMMA 2 (cf. [2], Theorem 2). *Any compact metric space X has an auxiliary inverse sequence $\{N(U_n);\ \pi_{m,n}\}$ associated with it and such that $\dim N(U_n) \leqslant \dim X$, and $\pi_{m,n}$ is a simplicial mapping of $N(U_m)$ onto a certain subdivision of $N(U_n)$ (for every m, n such that $m > n$).*

Proof. The proof is trivial in the case of $\dim X = 0$, and in the case of $\dim X > 0$ it suffices to apply the procedure of the proof of Theorem 2 from [2], applying our Factorization Lemma.

Combining Lemmas 1 and 2 we obtain the known Freudenthal Theorem.

The proof of our Theorem consists in applying the procedure from the proof of Theorem A from [2] together with Lemmas 1 and 2 and our Factorization Lemma. In order to get the additional conclusion in case (b), Remark 2 should be taken into consideration.

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Reçu par la Rédaction le 19. 6. 1972