COLLOQUIUM MATHEMATICUM

VOL. LVIII 1989 FASC. 1

CONVERGENCE OF POISSON INTEGRALS ON SEMIDIRECT EXTENSIONS OF HOMOGENEOUS GROUPS

BY

EWA DAMEK (WROCŁAW)

In this paper we continue to study harmonic functions with respect to leftinvariant hypoelliptic operators Lon a class of solvable Lie groups S. As in [4] our approach is motivated by the classical theory of harmonic functions on non-compact symmetric spaces.

A symmetric space $X = G/K$ may be identified with the group $\bar{N}A$, where $G = \bar{N}AK$ is the Iwasawa decomposition of G. After this identification the Laplace–Beltrami operator is a leftinvariant operator on $\bar{N}A$. For this reason the theory of harmonic functions is often formulated not in terms of G-homogeneous spaces but in terms of solvable Lie groups $\bar{N}A$ and their subgroups. Since the groups $\bar{N}$ have a natural structure of dilations, such an approach proved to be very useful in some cases and suggests to reformulate classical problems for leftinvariant operators on a larger class of spaces – solvable groups $S = NA$, which are semidirect products of homogeneous groups N and Abelian groups A.

The problem which has been of big interest in the case of symmetric spaces is the behavior of Poisson integrals $Pf(s)$, where s approaches a boundary of a compactification. The maximal compactification $\bar{X}$ of a symmetric space X consists of X and a finite number of boundaries. Each boundary is an infinite union of symmetric spaces of lower dimension called *boundary components*. One of them is the distinguished boundary $B = G/MAN$ which is compact and whose components are points. If $f \in L^1(B)$, the Poisson integral Pf has a natural extension $\bar{P}f$ to $\bar{X}$, which coincides with f on B. When f is continuous, so is $\bar{P}f$, and Pf converges to $\bar{P}f$ at the boundaries of X. For $f \in L^p(B)$, $p > 1$, we have the admissible convergence of Pf at almost all components of each boundary. This has been proved by Sjögren [14] and has finished more than 15 years' study of such problems ([10]–[12], [15]).

We have generalized the admissible convergence for the groups S and leftinvariant hypoelliptic operators on them. Although we do not use the language of compactifications, we can define some subgroups of S which correspond to boundaries and boundary components. Throughout this paper, S is a solvable Lie group such that

(i) the Lie algebra $\mathfrak{s}$ of S is a semidirect sum of a nilpotent algebra $\mathfrak{n}$ and an Abelian algebra $\mathfrak{a}$;

(ii) the operators $\mathrm{ad}_H|_{\mathfrak{n}}$, $H \in \mathfrak{a}$, are diagonalizable;

(iii) there is an $H \in \mathfrak{a}$ such that the operator $\mathrm{ad}_H|_{\mathfrak{n}}$ has strictly positive eigenvalues.

We study leftinvariant operators of the form

$$L = X_1^2 + \ldots + X_j^2 + X_0,$$

where $X_1, \ldots, X_j$, $X_0 \in \mathfrak{s}$ and $X_1, \ldots, X_j$ generate $\mathfrak{s}$. Bounded L-harmonic functions are exactly Poisson integrals Pf of L^∞-functions f defined on a subgroup $N_1(L)$ of N — the maximal ("distinguished") boundary for L (see [4]). We define decompositions $N = N_1 N_0$, $A = A_1 A_0$, where N_1, N_0, A, A_0 are Lie groups and $N_1 \subset N_1(L)$. The group $S_0 = N_0 A_0$ satisfies conditions (i), (ii), and (iii), so is of the same type as S. On S_0 we consider the operator L_0, which is a projection of L on S_0 defined in a very natural way. The group $N_0 \cap N_1(L)$ is the maximal ("distinguished") boundary for L_0, that is, L_0-harmonic bounded functions on S_0 are Poisson integrals of functions $f \in L^\infty(N_1(L) \cap N_0)$.

The groups $N_1 S_0$ for various N_1 correspond to boundaries of symmetric spaces and $x_1 S_0$, $x_1 \in N_1$, are "boundary components". We prove that if $f \in L^p(N_1(L))$, $p > 1$, then Pf converges to the Poisson integral $P_0 f$ on $N_1 S$ for almost all $x_1 \in N_1$ in the sense which generalizes the admissible convergence. Moreover, $P_0 f$ restricted to S_0 is an L_0-harmonic function. The proof relies on maximal function estimates and follows the ideas of Sjögren [14]. The estimates are possible because the kernels P, P_0 are smooth functions; they have a positive moment and decrease at infinity like a negative power of an invariant Riemannian distance.

The author is grateful to P. Sjögren for suggestion to consider kernels $P_z(y)$ (see (3.2)) as well as to A. Hulanicki for his generous assistance. The paper was written during her stay in Strasbourg and in Athens. She would like to express her gratitude to both hosts: Jacques Faraut and Ray Kunze.

1. Preliminaries. Let $\mathfrak{s}$ be a real solvable Lie algebra which is a semidirect sum

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$$

of a nilpotent algebra $\mathfrak{n}$ and an Abelian algebra $\mathfrak{a}$. We make the following assumptions about $\mathfrak{s}$:

(i) The operators $\mathrm{ad}_H|_{\mathfrak{n}}$, $H \in \mathfrak{a}$, are diagonal in a basis $E_1, \ldots, E_n$ of $\mathfrak{n}$, i.e.,

$$[H, E_i] = \lambda_i(H) E_i, \quad i = 1, \ldots, n,$$

for some $\lambda_i \in \mathfrak{a}^*$.

(ii) There is $H \in \mathfrak{a}$ such that

$$\lambda(H) > 0$$

for $H \in \mathfrak{a}$ and $\lambda \in \Delta = \{\lambda_1, \dots, \lambda_n\}$.

(iii) There is a basis of Δ over $\boldsymbol{R}$ such that every $\lambda \in \Delta$ is a linear combination of the elements of this basis with rational coefficients.

These assumptions are satisfied by solvable Lie algebras which appear in the Iwasawa decomposition of semisimple Lie algebras and, additionally, all roots $\lambda_1, \dots, \lambda_n$ are then linear combinations with integer non-negative coefficients of simple positive roots. Here we require only that the linear spaces spanned by Δ over the fields of rational and real numbers have the same dimension. A set of functionals having this property will be called *rational*. It is a technical assumption necessary for Sjögren's method and now we do not know how to omit it.

Let S, N, A be connected and simply connected Lie groups corresponding to the algebras $\mathfrak{s}$, $\mathfrak{n}$, $\mathfrak{a}$, respectively. S is a semidirect product of N and A: if

$$x = \exp\Big(\sum_{j=1}^{n} x_j E_j\Big) \in N,$$

then

$$axa^{-1} = x^a = \exp\Big(\sum_{j=1}^{n} \exp\big(\lambda_j(\log a)\big) x_j E_j\Big), \quad a \in A.$$

In particular, N is a homogeneous group. This immediately implies the following lemma:

(1.1) LEMMA ([4]). *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_0$ be the sum of its subalgebras $\mathfrak{n}_1$, $\mathfrak{n}_0$ invariant under the adjoint action of $\mathfrak{a}$ and let N_1, N_0 be the corresponding subgroups. Then*

$$I\colon N_1 \times N_0 \ni (y, z) \to yz \in N \tag{1.2}$$

is a diffeomorphism.

We will write $N = N_1 N_0$ if (1.2) holds and we call N_1, N_0 *homogeneous groups.*

Now we remind briefly basic facts about invariant Riemannian metrics on Lie groups which will be needed later. Let G be a connected Lie group with the rightinvariant Haar measure m. A non-negative Borel function ψ on G is called *subadditive* if it is bounded on compact sets and

(1.3) (i) $\psi(xy) \leqslant \psi(x) + \psi(y)$ for x, y in G, (ii) $\psi(x^{-1}) = \psi(x)$, $x \in G$.

If instead of (i) we have

$$\psi(xy) \leqslant \psi(x)\psi(y) \quad \text{for } x, y \in G$$

and also $\psi(x) \geqslant 1$, we say that ψ is *submultiplicative.*

Let $\| \ \|$ be a leftinvariant Riemannian metric on G and τ_G the corresponding distance (from the identity), i.e.,

$$\tau_G(x) = \inf \int_0^1 \|\dot{\sigma}(t)\|_{\sigma(t)} dt,$$

where the infimum is over all C^1-curves in G such that $\sigma(0) = e$, $\sigma(1) = x$ (cf., e.g., [8]), τ_G is subadditive and for every non-negative function ψ on G, which is bounded on compact sets and satisfies (1.3) (i), there is a constant C such that

$$\psi(x) \leqslant C(\tau_G(x)+1) \quad \text{for all } x \in G \tag{1.4}$$

(cf. Proposition 1.2 of [8]). Consequently, for every submultiplicative function ψ on G there is a constant C such that

$$\psi(x) \leqslant \exp\left(C(\tau_G(x)+1)\right), \quad x \in G. \tag{1.5}$$

If G' is a subgroup of G, by (1.4) we have

$$\tau_G(x) \leqslant C(\tau_{G'}(x)+1) \tag{1.6}$$

for a constant $C > 0$ and all $x \in G'$.

If $\alpha > 0$ is sufficiently large, then

$$\int_G \exp(-\alpha\tau_G(x)) dm(x) < \infty \tag{1.7}$$

(cf. [8]).

Let B be an automorphism of G and B^* its differential at the identity. For every $x \in G$

$$\tau_G(B(x)) \leqslant \|B^*\| \tau_G(x), \tag{1.8}$$

where $\|B^*\| = \sup\{\|B^*(w)\| : w \in T_e G, \ \|w\| = 1\}$ (see Proposition 1.8 of [4]).

Now we rewrite from [4] a few properties of Riemannian distances on N and its homogeneous subgroups. If $\| \ \|_1$ is a leftinvariant Riemannian metric on a homogeneous subgroup N_1 of N, and τ_{N_1} a corresponding distance, then there are positive C, β depending on N_1 such that

$$\|w\| \leqslant C(1+\tau_{N_1}(\exp w))^\beta \quad \text{for } w \in T_e N_1 \tag{1.9}$$

(cf., e.g., [4]).

The homogeneous structure of N yields here an inverse estimate.

(1.10) Lemma ([4]). *Let $N = N_1 N_0$ and τ_N, τ_{N_1}, τ_{N_0} be invariant Riemannian distances on N_1, N_0, N, respectively. Then there are $C > 0$, $\beta \geqslant 0$ such that*

$$\tau_{N_1}(y) + \tau_{N_0}(z) \leqslant C(1+\tau_N(yz))^\beta \tag{1.11}$$

for all y in N_1 and z in N_0.

Now, let $N = N_1 N_0$ and let τ_{N_1}, $\tau_{N_0 A}$, τ_S be arbitrary invariant Riemannian distances on N_1, N_0A and S. If, in addition, N_1 is a normal subgroup, then there is a constant C such that

$$\log(1+\tau_{N_1}(y))+\tau_{N_0A}(za) \leqslant C(\tau_S(s)+1), \tag{1.12}$$

where $y\in N_1$, $z\in N_0$, $a\in A$ and $s=yza$ (see [4] and [6]).

Finally, let us remark that since every nilpotent Lie group is of polynomial growth [5], there is a large α such that

$$\int_{N_1} (1+\tau_{N_1}(y))^{-\alpha}dy < \infty. \tag{1.13}$$

2. The main theorem. Let

$$L = X_1^2+ \ldots +X_j^2+X_0$$

be a leftinvariant operator on S such that $X_1,\ldots,X_j$ generate $\mathfrak{s}$, and so L is hypoelliptic. Let Z be the image of X in $\mathfrak{a}$ by the mapping

$$\mathfrak{s}\to\mathfrak{s}/\mathfrak{n} = \mathfrak{a}.$$

We define two sets

$$\Delta_1(L) = \{\lambda\in\Delta\colon\ \lambda(Z)<0\} \quad \text{and} \quad \Delta_0(L) = \Delta\backslash\Delta_1(L)$$

and two subalgebras

$$\mathfrak{n}_1(L) = \bigoplus_{\lambda\in\Delta_1(L)} \mathfrak{n}^\lambda \quad \text{and} \quad \mathfrak{n}_0(L) = \bigoplus_{\lambda\in\Delta_0(L)} \mathfrak{n}^\lambda,$$

where

$$\mathfrak{n}^\lambda = \{Y\in\mathfrak{n}\colon\ \forall_{H\in\mathfrak{a}}\ [H, Y] = \lambda(H)Y\}.$$

Then $\mathfrak{n} = \mathfrak{n}_1(L)\oplus\mathfrak{n}_0(L)$ and

$$N = N_1(L)N_0(L), \tag{2.1}$$

where $N_1(L) = \exp\mathfrak{n}_1(L)$ and $N_0(L) = \exp\mathfrak{n}_0(L)$.

Let Δ' be a subset of Δ including Δ_0 and such that if $\lambda,\eta\in\Delta'$ and $\lambda+\eta\in\Delta$, then $\lambda+\eta\in\Delta'$. We assume that for some Δ'

$$a(\Delta') = \{H\in\mathfrak{a}\colon\ \forall_{\lambda\in\Delta'}\ \lambda(H)=0,\ \forall_{\lambda\notin\Delta}\ \lambda(H)>0\} \neq 0. \tag{2.2}$$

If $\Delta_0 = \emptyset$, there is always a set Δ' such that (2.2) holds, because by assumption the set

$$\{H\colon\ \forall_{\lambda\in\Delta}\ \lambda(H)>0\}$$

is non-empty, and so is its boundary. Such a situation occurs for the Laplace–Beltrami operator on a symmetric space. If $\Delta_0 = \emptyset$ and all elements of Δ are non-negative combinations of a basis Π of Δ, then every set Δ' which satisfies (2.2) consists of non-negative combinations of a subset Π' of Π. If $\Delta_0 \neq \emptyset$

it may happen that there is no set Δ' satisfying (2.2). For example, let $\mathfrak{a} = \boldsymbol{R}^2$, λ_1, λ_2 be an arbitrary basis of $(\boldsymbol{R}^2)^*$, $\Delta = \{\lambda_1, \lambda_2, -\lambda_1+\lambda_2\}$, $\Delta_0 = \{\lambda_2\}$.

Let $\mathfrak{a}_1 \subset \mathfrak{a}$ be a subspace such that

$$(2.3)\qquad \begin{gathered}\text{(i) } \mathfrak{a}_1 \cap a^+(\Delta') \neq \emptyset; \quad \text{(ii) } \mathfrak{a}_1 \subset (\Delta')^\perp;\\ \text{(iii) } (\Delta\setminus\Delta')|_{\mathfrak{a}_1} \text{ is rational.}\end{gathered}$$

The following simple proposition shows that $\mathfrak{a}_1 = (\Delta')^\perp$ satisfies (2.3).

(2.4) PROPOSITION. *Let V be a linear space over $\boldsymbol{R}$, Δ a rational subset of V^*, and $\Delta' \subset \Delta$. Then $\Delta\setminus\Delta'|_{(\Delta')^\perp}$ is a rational subset of $\big((\Delta')^\perp\big)^*$.*

Let now $\mathfrak{a}$ be an arbitrary linear complement of $\mathfrak{a}_1$ in $\mathfrak{a}$,

$$A_1 = \exp\mathfrak{a}_1, \qquad A_0 = \exp\mathfrak{a}_0,$$
$$\mathfrak{n}_1 = \bigoplus_{\lambda\in\Delta\setminus\Delta'} \mathfrak{n}^\lambda, \qquad \mathfrak{n}_0 = \bigoplus_{\lambda\in\Delta'} \mathfrak{n}_0^\lambda.$$

Then $\mathfrak{n}_1$, $\mathfrak{n}_0$ are subalgebras, $N_1 = \exp\mathfrak{n}_1$, $N_0 = \exp\mathfrak{n}_0$ are homogeneous subgroups, and $N = N_1N_0$. Moreover, N_1 is a normal subgroup because $\lambda+\eta\notin\Delta'$ if $\lambda\notin\Delta'$ and η, $\lambda+\eta\in\Delta$. Since by (2.3) elements of A_1 and N_0 commute, N_1A_1 is a normal subgroup of S and

$$S = N_1N_0A_1A_0 = N_1A_1S_0, \qquad \text{where } S_0 = N_0A_0.$$

Let $s = x_1x_0a_1a_0$ with $x_1\in N_1$, $x_0\in N_0$, $a_1\in A_1$, $a_0\in A_0$. We define projections $\Pi_{N_1}\colon S\to N_1$ and $\Pi_{S_0}\colon S\to S_0$ by

$$\Pi_{N_1}(s) = x_1 \qquad \text{and} \qquad \Pi_{S_0}(s) = x_0a_0.$$

The closure $\bar{L}$ of L (restricted to $C_c^\infty(S)$) in C_0 is the infinitesimal generator of a semigroup of probability measures $\{\mu_t\}_{t>0}$ (cf., e.g., [9]). Since Π_{S_0} is a homomorphism, $\{\Pi_{S_0}(\mu_t)\}$ is a semigroup on S_0 with the infinitesimal generator $\bar{L}_0$, where

$$L_0 = (X_1')^2+\ldots+(X_j')^2+X_0',$$

and X_i', $i = 0,\ldots,j$, is the projection of X_i on $\mathfrak{s}_0 = \mathfrak{s}/(\mathfrak{n}_1\oplus\mathfrak{a}_1)$. $X_1',\ldots,X_j'$ generate $\mathfrak{s}_0$, and L_0 is also a hypoelliptic operator.

The group S_0 is of the same type as S. Its algebra $\mathfrak{s}_0$ is a semidirect sum of $\mathfrak{n}_0$ and $\mathfrak{a}_0$, the adjoint action of $\mathfrak{a}_0$ on $\mathfrak{n}$ is diagonal. If $\lambda(H) > 0$ for $\lambda\in\Delta$ and $H = H_1+H_0$, $H_1\in\mathfrak{a}_1$, $H_0\in\mathfrak{a}_0$, then $\lambda(H_0) = \lambda(H)$ for $\lambda\in\Delta'$, and hence $\mathrm{ad}_{H_0}|_{\mathfrak{n}_0}$ has positive eigenvalues.

Let $Z = Z_1+Z_0$, $Z_1\in\mathfrak{a}_1$, $Z_0\in\mathfrak{a}_0$. Z_0 is the projection of X_0' onto $\mathfrak{a}_0 = \mathfrak{s}_0/\mathfrak{n}_0$, $\lambda(Z) = \lambda(Z_0)$ for $\lambda\in\Delta'$, and so

$$\{\lambda\in\Delta'\colon \lambda(Z_0) < 0\} = \Delta'\setminus\Delta_0.$$

Therefore

$$\mathfrak{n}_1(L_0) = \bigoplus_{\lambda\in\Delta'\setminus\Delta_0} \mathfrak{n}^\lambda, \qquad \mathfrak{n}_0(L_0) = \mathfrak{n}_0(L).$$

Let $N_2 = \exp \mathfrak{n}_1(L_0)$. Then $N_1(L) = N_1 N_0$ and $N_0 = N_2 N_0(L)$.

We study L-harmonic functions on S and L_0-harmonic functions on S_0, that is, the functions F such that $LF = 0$ or $L_0 F = 0$, respectively. It has been proved in [4] that there is an integrable, smooth, bounded function P on $N_1(L)$ called a *Poisson kernel* such that bounded L-harmonic functions on N are precisely Poisson integrals

$$(2.5)\qquad Pf(s) = \int_{N_1(L)} f(\Pi_{N_1(L)}(su))P(u)du,$$

where $f \in L^\infty(N_1(L))$ and $\Pi_{N_1(L)}(s) = u_1$ if $s = u_1 u_0 a$, $u_1 \in N_1(L)$, $u_0 \in N_0(L)$, $a \in A$. Similarly, if $\Delta' \neq \Delta_0$, then on N_2 we have an integrable, smooth, bounded kernel P_0 and Poisson integrals

$$P_0 g(s_0) = \int_{N_2} g(\Pi_{N_2}(s_0 z))P_0(z)dz,$$

where $g \in L^\infty(N_2)$ and $\Pi_{N_2}(s) = z$ if $s = yzu_0 a$, $y \in N_1$, $z \in N_2$, $u_0 \in N_0(L)$, $a \in A$. Every bounded L_0-harmonic function on S_0 is equal to $P_0 g$ for a $g \in L^\infty(N_2)$. If $\Delta_0 = \Delta'$ and $N_2 = \emptyset$, then by a theorem of Birgé and Raugi [2] there are no bounded harmonic functions on S_0.

L-harmonic bounded functions F are, in particular, $\check{\mu}_1$-harmonic in the sense of Azencott and Cartier [1], Guivarc'h [7], and Raugi [13]; hence

$$(2.6)\qquad F(s) = \int_S F(ss')d\check{\mu}_1(s').$$

Analogously, an L_0-harmonic bounded function G on S_0 satisfies

$$G(s_0) = \int_{S_0} G(ss_0')d\Pi_{S_0}(\check{\mu}_1)(s_0')$$

because $(\Pi_{S_0}(\mu_1))^\vee = \Pi_{S_0}(\check{\mu}_1)$. By (2.15) of [4], P and P_0 are *weak limits of

$$\Pi_{N_1(L)}((\check{\mu}_1)^{*n}) \quad \text{and} \quad \Pi_{N_2}(\Pi_{S_0}(\check{\mu}_1)^{*n}),$$

respectively, and so

$$(2.7)\qquad P_0(z) = \int_{N_1} P(yz)dy.$$

If $f \in L^p(N_1(L))$, Pf given by (2.5) is an L-harmonic function in view of Theorem 3.17 of [4]. We define now integrals $P_0 f$ for $f \in L^p(N_1(L))$. If $N_2 \neq \emptyset$ and $s = x_1 x_0 a_1 a_0$, $x_1 \in N_1$, $x_0 \in N_0$, $a_1 \in A_1$, $a_0 \in A_0$, we write

$$(2.8)\qquad P_0 f(s) = \int_{N_2} f(x_1 \Pi_{N_2}(x_0 a_0 z))P_0(z)dz.$$

$P_0 f$ is defined for almost all s, and if x_1 is fixed, $P_0 f$ is L_0-harmonic as a function of $x_0 a_0$ (Theorem 3.17 of [4]). If $N_2 = \emptyset$, then

$$P_0 f(s) = f(\Pi_{N_1(L)}(s)).$$

Let $a_1 \in \mathfrak{a}_1$. We say that $\log a_1 \to -\infty$ if $\lambda(\log a_1) \to -\infty$ for all $\lambda \in \Delta \setminus \Delta'$.

Conditions (2.3) guarantee that this definition makes sense.

Now we are ready to formulate our main theorem.

(2.9) THEOREM. *Let $K_1 \subset N_1$ and $K_0 \subset S_0$ be compact sets and $f \in L^p(N_1(L))$ for a $p > 1$. If $\log a_1 \to -\infty$, $a_1 \in A_1$, then for a.e. $s \in S$*

$$Pf(sa_1 ys_0) \to P_0 f(ss_0) \tag{2.10}$$

uniformly with respect to $y \in K_1$, $s_0 \in K_0$.

Remarks. 1. "Uniformly" means that there is a measurable set $M \subset N_1$ with the complement of the Lebesgue measure 0 satisfying the following condition:

For all $x_1 \in M$, $\varepsilon > 0$ and any compact sets $K_1 \subset N_1$, $K_0 \subset S_0$ we can find $a'_1 \in A_1$ such that

$$|Pf(sa_1 ys_0) - P_0 f(ss_0)| < \varepsilon$$

whenever $y \in K_1$, $s_0 \in K_0$ and $\lambda(a_1) \leqslant \lambda(a'_1)$ for all $\lambda \in \Delta \backslash \Delta'$.

2. Theorem (2.9) is a generalization of Theorem (4.1)a of [14]. Although we do not construct a compactification of S, we define spaces which correspond to boundaries and boundary components of the maximal compactification of a symmetric space. The group $N_1 S_0$ corresponds to a boundary and $x_1 S_0$, $x_1 \in N_1$, are "boundary components" diffeomorphic to S_0, which is a solvable Lie group of the same type as S but of a lower dimension.

3. In the case of a symmetric space with the set of simple positive roots Π, $\mathfrak{a}_1$ is the annihilator of $\Pi' \subset \Pi$. Since we do not assume that the elements of Δ are positive combinations of a basis of Δ, we only require for the space $\mathfrak{a}_1$ to have the weakest properties necessary to formulate Theorem (2.9). They are included in (2.3) and

(i) guarantees that we can define a convergence to $-\infty$ in $\mathfrak{a}_1$;

(ii) implies that N_1 is included in the maximal boundary $N_1(L)$, and if $a_1 \in A_1$, $x_0 \in N_0$, then $a_1 x_0 a_1^{-1} = x_0$;

(iii) is necessary to apply P. Sjögren's method and we do not know whether it is possible to remove it.

Proof of Theorem (2.9). Obviously, it is sufficient to prove the theorem for $s \in N_1$. Since N_1 is a normal subgroup and $a_1 x_0 = x_0 a_1$ for $x_0 \in N_0$, $a_1 \in A_1$, we have

$$Pf(x_1 ay_1 s_0) = \int_{N_1 \times N_2} f(x_1 \Pi_{N_1}(ay_1 s_0 y) \Pi_{N_2}(s_0 z)) P(yz) dy dz$$

and $\Pi_{N_1}(ay_1 s_0 y) = ay_1 s_0 y s_0^{-1} a^{-1}$, where $x_1, y_1 \in N_1$, $a \in A_1$, $s_0 \in S_0$. Therefore, if $\log a \to -\infty$, then $\Pi_{N_1}(ay_1 s_0 y)$ converges to the identity element of N_1 and

$$Pfx_1 ay_1 s_0) \to P_0 f(x_1 s_0)$$

for a bounded continuous f.

Let

$$R_{K_1,K_0}f(x_1) = \sup\{|Pf(x_1ay_1s_0)-P_0f(x_1s_0)|:\ a\in A_1,\ y_1\in K_1,\ s_0\in K_0\}.$$

We shall prove that, for every $p > 1$, R_{K_1,K_0}: $L^p(N_1(L))\to L^p(N_1)$ is a bounded operator, and then the theorem will follow by a standard approximation argument.

Since

$$R_{K_1,K_0}f(x_1) \leqslant M'f(x_1)+M''f(x_1),$$

where

$$M'f(x_1) = \sup\{|Pf(x_1ay_1s_0)|:\ a\in A_1,\ y_1\in K_1,\ s_0\in K_0\},$$
$$M''f(x_1) = \sup_{s_0\in S_0}|P_0f(x_1s_0)|,$$

it is sufficient to prove that M', M'': $L^p(N_1(L))\to L^p(N_1)$ are bounded operators, which is done in the next two theorems.

(2.11) THEOREM. *Let $f\in L^p(N_1(L))$ for a $p\geqslant 1$ and let K_0 be a compact set in S_0. Then*

$$M'':\ L^p(N_1(L))\to L^p(N_1)$$

defined by

$$M''f(x_1) = \sup_{s_0\in S_0}|P_0f(x_1s_0)|$$

is a bounded operator.

Proof. Let

$$F_{x_1}(s_0) = \int_{N_2}|f(x_1\Pi_{N_2}(s_0z))|P_0(z)dz.$$

F_{x_1} is finite for almost all $x_1\in N_1$, and then L_0-harmonic as a function on S_0. A Harnack inequality [3] implies that there is $C_{K_0} > 0$ such that if $s_0\in K_0$, then

$$F_{x_1}(s_0) \leqslant C_{K_0}F_{x_1}(e)$$

for almost all $x_1\in N_1$. Let $p^{-1}+q^{-1} = 1$. Since $P_0\in L^q$, $q\geqslant 1$, we have

$$M''f(x_1) \leqslant C_{K_0}\int_{N_2}|f(x_1z)|P_0(z)dz$$
$$\leqslant C_{K_0}\|P_0\|_{L^q(N_2)}\Big(\int_{N_2}|f(x_1z)|^p dz\Big)^{1/p}.$$

Therefore

$$\|M''f\|_{L^p(N_1)} \leqslant C_{K_0}\|P_0\|_{L^q(N_2)}\|f\|_{L^p(N_1(L))}.$$

(2.12) THEOREM. *Let $f\in L^p(N_1(L))$ for a $p>1$ and let $K_1\subset N_1$, $K_0\subset S_0$ be compact sets. If*

$$M'f(x_1) = \sup\{|Pf(x_1ay_1s_0)|:\ a\in A_1,\ y_1\in K_1,\ s_0\in K_0\},$$

then there is a constant $C_p > 0$ *such that for every* $f \in L^p(N_1(L))$

$$\|M'f\|_{L^p(N_1)} \leqslant C_p \|f\|_{L^p(N_1(L))}. \tag{2.13}$$

Proof. In view of (2.3) we can choose a basis $H_1, \ldots, H_l$ of $\mathfrak{a}_1$ such that $\lambda(H_j)$ is integer for $\lambda \in \Delta \backslash \Delta'$, $j = 1, \ldots, l$. We write

$$[a] = \exp\left(\sum_{i=1}^{l} [a_i] H_i\right) \quad \text{if } a = \exp\left(\sum_{i=1}^{l} a_i H_i\right).$$

Let

$$K = \{a y_1 s_0 \colon \ -1 \leqslant a_i \leqslant 1,\ y_1 \in K_1,\ s_0 \in K_0\}.$$

Since K is a compact set and L is leftinvariant, a Harnack inequality [3] implies that there is a constant C_K such that

$$P|f|(x_1 a y_1 s_0) \leqslant C_K P|f|(x_1 [a]).$$

Let $Y_1, \ldots, Y_\chi$ be a basis of $\mathfrak{n}_1$ consisting of eigenvectors for the action of $\mathfrak{a}$, that is, $[H, Y_j] = \lambda_{i_j}(H) Y_j$ for $H \in \mathfrak{a}$ and $\lambda_{i_1}, \ldots, \lambda_{i_\chi}$ are ordered so that if $\lambda_{i_j} + \lambda_{i_k} = \lambda_{i_r}$, $1 \leqslant j,\ k,\ r \leqslant \chi$, then $r > \max(j, k)$. Then $\mathrm{lin}(Y_i, \ldots, Y_\chi)$ is an ideal of $\mathfrak{n}_1$, $i = 1, \ldots, \chi$, and therefore every $y \in N_1$ can be written as

$$y = \prod_{i=1}^{\chi} \exp(y_i Y_i) \tag{2.14}$$

(cf., e.g., [14]). Hence

$$[a] y [a]^{-1} = \prod_{i=1}^{\chi} \exp(y_j \exp(\lambda_{i_j}(\log[a])) Y_j) \tag{2.15}$$

and $\lambda_{i_j}(\log[a])$ is integer for $j = 1, \ldots, \chi$. Since $z[a]^{-1} = [a]^{-1} z$, in view of (2.15) we have

$$\begin{aligned} P|f|(x_1[a]) &= \int_{N_1 \times N_2} |f(x_1 [a] y [a]^{-1} z)| P(yz)\, dy\, dz \\ &\leqslant \sup_{h \in Z^\chi} \int_{N_1 \times N_2} |f(x_1 \delta_h(y) z)| P(yz)\, dy\, dz = Mf(x_1), \end{aligned} \tag{2.16}$$

where $h = (h_1, \ldots, h_\chi) \in Z^\chi$ and

$$\delta_h\left(\prod_{i=1}^{\chi} \exp(y_i Y_i)\right) = \prod_{i=1}^{\chi} \exp(y_i e^{h_i} Y_i). \tag{2.17}$$

Let now $g \in L^p(N_1)$ for a $p > 1$. For every $z \in N_2$, $x_1 \in N_1$ we define

$$M_z g(x_1) = \sup_{h \in Z^\chi} \int_{N_1} g(x_1 \delta_h(y)) P(yz)\, dy.$$

In the next section we shall prove that for a fixed p there are $C, \alpha > 0$ such that for every $z \in N_2$ and every $g \in L^p(N_1)$

$$\|M_z g\|_{L^p(N_1)} \leqslant C(1 + \tau_{N_2}(z))^{-\alpha} \|g\|_{L^p(N_1)}. \tag{2.18}$$

Now we are completing our proof assuming (2.18). By (1.13) there is q_0 such that for $q > q_0$

$$b_q^q = \int_{N_2} (1+\tau_{N_2}(z))^{-\alpha q} dz < \infty.$$

Let $q > q_0$, $q^{-1}+p^{-1} = 1$ and $f \in L^p(N_1(L))$. The function $f_z(y) = f(yz)$ belongs to $L^p(N_1)$ for almost all $z \in N_2$. Then $M_z f_z$ is defined and

$$Mf(x_1) \leqslant \int_{N_2} M_z f_z(x_1) dz \leqslant b_q \Big(\int_{N_2} (1+\tau_{N_2}(z))^{\alpha p} |M_z f_z(x_1)|^p dz\Big)^{1/p}$$

for a.e. $x_1 \in N_1$. Hence, by (2.18),

$$\|Mf\|_{L^p(N_1)} \leqslant C b_q \|f\|_{L^p(N_1(L))},$$

which in view of (2.16) gives (2.13) for $p \leqslant p_0$, where $p_0^{-1}+q_0^{-1} = 1$. But for $p = \infty$ (2.13) is obvious, which yields the assertion.

3. A maximal function. This section is devoted to a proof of the following theorem:

(3.1) THEOREM. *Let $f \in L^p(N_1)$ for a $p > 1$, $x \in N_1$, $z \in N_2$, let δ_h be as in* (2.17) *and*

$$M_z f(x) = \sup_{h \in Z^\chi} \int_{N_1} f(x\delta_h(y)) P(yx) dy.$$

Then there are C, $\alpha > 0$ such that for every $z \in N_2$ and every $f \in L^p(N_1)$

$$\|M_z f\|_{L^p(N_1)} \leqslant C(1+\tau_{N_2}(z))^{-\alpha} \|f\|_{L^p(N_1)}.$$

First we are going to remind a few properties of P proved in [4] and to derive analogous ones of kernels P_z, where

$$P_z(y) = P(yz). \tag{3.2}$$

Properties of the kernel P:

There are C, $\varepsilon > 0$ such that

$$P(u) \leqslant C(1+\tau_{N_1(L)}(u))^{-\varepsilon} \tag{3.3}$$

(Theorem 4.4 of [4]);

$$Y' P(u) = \frac{d}{dt} P(\exp tY \cdot u)|_{t=0}, \qquad Y \in \mathfrak{n}_1(L), \tag{3.4}$$

is a bounded function on $N_1(L)$ (Theorem 3.15 of [4]);

$$\int_{N_1(L)} (1+\tau_{N_1(L)}(u))^\eta P(u) du < \infty \tag{3.5}$$

for some $\eta > 0$ (Theorem 3.10 of [4]).

Let $Y_1, \ldots, Y_\chi$ be a basis of $\mathfrak{n}_1$, and $y_1, \ldots, y_\chi$ coordinates on N_1 introduced in (2.14).

(3.6) LEMMA. *There are C, γ, $\varphi > 0$ such that for every z*

$$P_z(y) \leqslant C\min(1, |y_1|^{-\gamma}, \ldots, |y_\chi|^{-\gamma})(1+\tau_{N_2}(z))^{-\varphi}.$$

Proof. The lemma follows by (3.3), Lemma (1.10) and the inequality

$$\max_{i=1,\ldots,\chi} |y_i| \leqslant C(1+\tau_{N_1}(y))^{\xi}$$

for some C, $\xi > 0$ (cf., e.g., Proposition 1.10 of [4]).

The following lemma is immediately implied by (3.4).

(3.7) LEMMA. *Let*

$$Y_j' f(y) = \frac{d}{dt} f(\exp tY_j \cdot y)|_{t=0}, \quad y \in N_1.$$

Then there is $C > 0$ such that for every $z \in N_2$ and $j = 1, \ldots, \chi$ we have

$$\| Y_j' P_z \|_{L^\infty(N_1)} \leqslant C.$$

Applying Lemma (3.7) in a similar way as in the proof of Lemma 4.9 of [4] we obtain

(3.8) LEMMA. *There are C, $\sigma > 0$ such that for every $z \in N_2$ and every y, $u \in N_1$*

$$|P_z(y) - P_z(u)| \leqslant C \|y-u\| (1 + \|y-u\| + \|y\|)^{\sigma},$$

where

$$\|y\| = \left(\sum_{i=1}^{\chi} |y_i|^2\right)^{1/2}.$$

Now we fix leftinvariant distances τ_N, τ_{N_1}, τ_{N_2}, τ_{N_0}, $\tau_{N_0(L)}$, τ_S, τ_A on the corresponding groups. By Lemma (1.10) there are

$$\beta_1, \beta_2, \beta_3, C_1, C_2, C_3 > 0$$

such that

$$\tau_{N_1}(y) \leqslant C_1(1+\tau_N(y))^{\beta_1} \quad \text{for } y \in N_1, \tag{3.9}$$

$$\tau_{N_2}(z) \leqslant C_2(1+\tau_{N_0}(zw))^{\beta_2} \quad \text{for } z \in N_2,\ w \in N_0(L), \tag{3.10}$$

$$(1+\tau_{N_2}(z))(1+\tau_{N_0(L)}(w)) \leqslant C_3(1+\tau_{N_0}(zw))^{\beta_3} \quad \text{for } z \in N_2,\ w \in N_0(L). \tag{3.11}$$

Let q be the density of $\check{\mu}_1$ with respect to a rightinvariant measure m on S. Since N_1, N are normal subgroups of S, we have

$$\check{\mu} = q(yxa)\,dy\,dx\,da,$$

where dy, dx, da are Haar measures on N_1, $N_0 = N_2 N_0(L)$, A, respectively. Let $\Delta' = \{\lambda_1, \ldots, \lambda_m\}$ and $E_1, \ldots, E_m$ be a basis of $\mathfrak{n}_0$ consisting of eigenvectors, i.e.,

$$[H, E_j] = \lambda_j(H) E_j \quad \text{for } H \in \mathfrak{a}.$$

We can introduce an ordering of the set Δ' satisfying $\lambda_i+\lambda_j > \lambda_i$ whenever $\lambda_i+\lambda_j \in \Delta'$ (see [4]) and assume that

$$\lambda_1 \leqslant \ldots \leqslant \lambda_m.$$

If coordinates in N_0 are given by

$$x = \exp(x_1 E_1 + \ldots + x_m E_m),$$

then

$$(3.12) \qquad (xx')_i = x_i + x_i' + T_i(x_1, \ldots, x_{i-1}, x_1', \ldots, x_{i-1}'),$$

where T_i are polynomials. Since either $E_i \in \mathfrak{n}_2$ or $E_i \in \mathfrak{n}_0(L)$, (3.12) implies that the mapping

$$N_2 \times N_0(L) \ni (z, w) \to zw \in N_0$$

is a diffeomorphism whose differential has determinant 1. Therefore, if dz and dw are Haar measures on N_2 and $N_0(L)$, respectively, then $dydzdwda$ is a rightinvariant measure on S and

$$\check{\mu} = q(yzwa)dydzdwda.$$

(3.13) LEMMA. *For every $\alpha > 0$ there is $C > 0$ such that*

$$(3.14) \quad q(yzwa) \leqslant C(1+\tau_{N_1}(y))^{-\alpha}(1+\tau_{N_2}(z))^{-\alpha}(1+\tau_{N_0(L)}(w))^{-\alpha}\exp(-\alpha\tau_A(a)).$$

Proof. Since N_1 is a normal subgroup in N, and N_0 a normal subgroup in S_0, applying twice (1.12) we obtain

$$(3.15) \qquad \tau_S(yzwa) \geqslant C'(\log(1+\tau_{N_1}(y)) + \log(1+\tau_{N_0}(zw)) + \tau_A(a)) - C''$$

for some constants C', $C'' > 0$. Let β_3 be as in (3.11) and

$$\alpha' = \max(\alpha(C')^{-1}, (\alpha\beta_3)(C')^{-1}).$$

By Proposition 1.21 of [4] there is $C = C(\alpha') > 0$ such that

$$q(yzwa) \leqslant C\exp(-\alpha'\tau_S(yzwa)).$$

This combined with (3.15) and (3.10) yields (3.14).

(3.16) THEOREM. *There are C, σ, $\eta > 0$ such that for a.e. $z \in N_2$*

$$(3.17) \qquad \int_{N_1} P_z(y)(1+\tau_{N_1}(y))^{\eta}dy \leqslant C(1+\tau_{N_2}(z))^{-\sigma}.$$

Proof. Let $Y_1, \ldots, Y_\chi$ be a basis of $\mathfrak{n}_1$ and $Y_{\chi+1}, \ldots, Y_\xi$ a basis of $\mathfrak{n}_2$ such that

$$[H, Y_j] = \lambda_{i_j}(H)Y_j, \qquad H \in \mathfrak{a}.$$

We write

$$D_1(a) = \exp\Big(\sum_{j=1}^{\chi} \lambda_{i_j}(\log a)\Big), \qquad D_2(a) = \exp\Big(\sum_{j=\chi+1}^{\xi} \lambda_{i_j}(\log a)\Big),$$

and $D(a)=D_1(a)D_2(a)$. If

$$P(s,u)=P(\Pi_{N_1(L)}(s^{-1}u))D(a^{-1}), \tag{3.18}$$

then

$$Pf(s)=\int_{N_1(L)} f(u)P(s,u)\,du. \tag{3.19}$$

Since $P\in C^\infty(S\times N_1(L))$, (3.19), (2.6) and the Torelli theorem yield

$$P(s,e)=\int_S P(ss',e)\,d\check{\mu}_1(s'). \tag{3.20}$$

By (3.5) and (1.11) there is $\varepsilon>0$ such that

$$\int_{N_1\times N_2} (1+\tau_{N_1}(y))^\varepsilon(1+\tau_{N_2}(z))^\varepsilon P(yz)\,dy\,dz<\infty, \tag{3.21}$$

and so the function

$$R(z)=\int_{N_1}(1+\tau_{N_1}(y))^\varepsilon P(yz)\,dy$$

is finite for a.e. $z\in N_2$.

Let now $\bar{P}(s)=P(s,e)$. If $u\in N_1(L)$, then $\bar{P}(u)=P(u^{-1})$. For

$$\gamma\leqslant\min(\varepsilon\beta_2^{-2},\ \varepsilon\beta_1^{-1}\beta_2^{-1},\ \tfrac{1}{3}\varepsilon\beta_1^{-1})$$

we define

$$R'(s)=\int_{N_1}(1+\tau_{N_1}(y))^\gamma\bar{P}(ys)\,dy.$$

We shall show that, for almost all $s\in S$, $R'(s)$ is finite. If $s=uzwa$, $u\in N_1$, $z\in N_2$, $w\in N_0(L)$, $a\in A$, then by (1.3) (i) we have

$$\begin{aligned}R'(s)&=\int_{N_1}(1+\tau_{N_1}(y))^\gamma\bar{P}(yuzwa)\,dy\\&\leqslant(1+\tau_{N_1}(u))^\gamma\int_{N_1}(1+\tau_{N_1}(y))^\gamma\bar{P}(yzwa)\,dy.\end{aligned}$$

But, by (3.18),

$$\bar{P}(yzwa)=P(a^{-1}w^{-1}z^{-1}y^{-1}zw\Pi_{N_2}(w^{-1}z^{-1})a)D(a^{-1})$$

because N_1 is a normal subgroup. Therefore

$$\begin{aligned}R'(s)\leqslant(1+\tau_{N_1}(u))^\gamma&\int_{N_1}(1+\tau_{N_1}(zwaya^{-1}w^{-1}z^{-1}))^\gamma\\&\times P(ya^{-1}\Pi_{N_2}(w^{-1}z^{-1})a)\,dy\cdot D_2(a^{-1}).\end{aligned}$$

Applying first (1.11), and then (1.6), we see that

$$(1+\tau_{N_1}(zwaya^{-1}w^{-1}z^{-1}))^\gamma\leqslant C(1+\tau_{N_1}(aya^{-1}))^{\gamma_1}(1+\tau_{N_2}(z))^{\gamma_1}(1+\tau_{N_0(L)}(w))^{\gamma_1}$$

for a constant $C > 0$ and $\gamma_1 = \gamma\beta_1$, where β_1 is as in (3.9). Now (1.8) implies that

$$R'(s) \leqslant C(1+\tau_{N_1}(u))^{\gamma}(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\gamma_1}(1+\tau_{N_2}(z))^{\gamma_1} \times (1+\tau_{N_0(L)}(w))^{\gamma_1} D_2(a^{-1}) R(a^{-1}\Pi_{N_2}(w^{-1}z^{-1})a), \tag{3.22}$$

which proves that $R'(s)$ is finite for almost all $s \in S$. By (3.20) and the Fubini theorem we have

$$R'(s) = \int_S R'(ss_1)d\check{\mu}_1(s_1). \tag{3.23}$$

We use (3.23) to prove that there are C, $\delta > 0$ such that

$$R'(v) \leqslant C(1+\tau_{N_2}(v))^{-\delta}. \tag{3.24}$$

Let $v \in N_2$ and $\delta \leqslant \frac{1}{3}\varepsilon\beta_2^{-1}$. By (3.14), (3.22) and (3.23) for every $\alpha > 0$ we have

$$\begin{aligned} R'(v)(1+\tau_{N_2}(v))^{\delta} &= \int_S (1+\tau_{N_2}(v))^{\delta} R'(vyzwa)\, q(yzwa)dydzdwda \\ &\leqslant C(\alpha)\int_S (1+\tau_{N_2}(v))^{\delta}(1+\tau_{N_1}(vyv^{-1}))^{\gamma}(1+\tau_{N_1}(y))^{-\alpha} \\ &\quad \times (1+\tau_{N_2}(vz))^{\gamma_1}(1+\tau_{N_2}(z))^{-\alpha}(1+\tau_{N_0(L)}(w))^{\gamma_1-\alpha} \\ &\quad \times D_2(a^{-1})(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\gamma_1}\exp(-\alpha\tau_A(a)) \\ &\quad \times R(a^{-1}\Pi_{N_2}(w^{-1}z^{-1}v^{-1})a)dydzdwda. \end{aligned}$$

Let

$$\psi(a) = D_2(a^{-1})(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\gamma_1}\exp(-\alpha\tau_A(a)).$$

We have

$$(1+\tau_{N_2}(v))^{\delta+\gamma_1} \leqslant (1+\tau_{N_2}(vz))^{\delta+\gamma_1}(1+\tau_{N_2}(z))^{\delta+\gamma_1}$$

and, by (3.9),

$$(1+\tau_{N_1}(vyv^{-1}))^{\gamma} \leqslant C(1+\tau_{N_2}(v))^{\gamma_1}(1+\tau_{N_1}(y))^{\gamma_1}.$$

Hence

$$\begin{aligned} R'(v)(1+\tau_{N_2}(v))^{\delta} &\leqslant C\int_S (1+\tau_{N_1}(y))^{\gamma_1-\alpha}\ (1+\tau_{N_2}(vz))^{\delta+2\gamma_1} \\ &\quad \times (1+\tau_{N_2}(z))^{\delta+\gamma_1-\alpha}(1+\tau_{N_0(L)}(w))^{\gamma_1-\alpha}\psi(a) \\ &\quad \times R(a^{-1}\Pi_{N_2}(w^{-1}z^{-1}v^{-1})a)dydzdwda. \end{aligned}$$

Let

$$\psi(y, a) = (1+\tau_{N_1}(y))^{\gamma_1-\alpha}\psi(a).$$

Since $(1+\tau_{N_2}(z))^{\delta+\gamma_1-\alpha} < 1$ for α sufficiently large, so integrating first over z we obtain

$$R'(v)(1+\tau_{N_2}(v))^{\delta}$$
$$\leqslant \int_S (1+\tau_{N_2}(z))^{\delta+2\gamma_1} R(a^{-1}\Pi_{N_2}(w^{-1}z)a)(1+\tau_{N_0(L)}(w))^{\gamma_1-\alpha}\psi(y,a)dzdydwda$$
$$= \int_S (1+\tau_{N_2}(\Pi_{N_2}(wz)))^{\delta+2\gamma_1} R(a^{-1}za)(1+\tau_{N_0(L)}(w))^{\gamma_1-\alpha}\psi(y,a)dzdydwda.$$

But, in view of (3.10),

$$\tau_{N_2}(\Pi_{N_2}(wz)) \leqslant C(1+\tau_{N_2}(z))^{\beta_2}(1+\tau_{N_0(L)}(w))^{\beta_2},$$

hence, by (1.8),

$$(3.25) \quad R'(v)(1+\tau_{N_2}(v))^{\gamma}$$
$$\leqslant C\int (1+\tau_{N_2}(z))^{\beta_2(\delta+2\gamma_1)} R(z)(1+\tau_{N_0(L)}(w))^{\beta_2(\delta+2\gamma_1)+\gamma_1-\alpha} D_2(a)$$
$$\times (1+\|\mathrm{Ad}_a|_{\mathfrak{n}_2}\|)^{\beta_2(\delta+2\gamma_1)}\psi(y,a)dzdydwda.$$

Since $a \to (1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\gamma_1}$ and $a \to (1+\|\mathrm{Ad}_a|_{\mathfrak{n}_2}\|)^{\beta_2(\delta+2\gamma_1)}$ are submultiplicative functions, by (1.5) and (1.7) we have

$$\int_S (1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\gamma_1}(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_2}\|)^{\beta_2(\delta+2\gamma_1)}\exp(-\alpha\tau(a))da < \infty$$

for α large enough. This together with (1.13) and (3.21) implies that for such α the integral on the right-hand side of (3.25) is finite. Now, let

$$\eta \leqslant \min(\gamma\beta_1^{-1}, \tfrac{1}{2}\delta\beta_1^{-1})$$

and let δ be as in (3.24). Then

$$\int_{N_1} (1+\tau_{N_1}(y))^{\eta} P(yz)dy = \int_{N_1} (1+\tau_{N_1}(zyz^{-1}))^{\eta} P(zy)dy$$
$$\leqslant C\Big(\int_{N_1} (1+\tau_{N_1}(y))^{\beta_1\eta} P(zy^{-1})dy\Big)(1+\tau_{N_2}(z))^{\beta_1\eta}$$
$$\leqslant CR'(z^{-1})(1+\tau_{N_2}(z))^{\beta_1\eta} \leqslant C(1+\tau_{N_2}(z))^{-\delta/2},$$

which completes the proof.

(3.26) LEMMA. *There are C, α, $\varepsilon > 0$, $\varepsilon < 1$, such that for almost all $z \in N_2$*

$$(3.27) \quad \int_{N_1} P_z(y)^{1-\varepsilon}dy \leqslant C(1+\tau_{N_2}(z))^{-\alpha}.$$

Proof. Let η be as in the previous theorem and γ such that

$$C_1 = \int_{N_1} (1+\tau_{N_1}(y))^{-\gamma}dy < \infty.$$

If $\varepsilon = \eta/(\gamma+\eta)$ and $\delta = \eta\gamma/(\gamma+\eta)$, then by the Hölder theorem and Theorem (3.16) we have

$$\int_{N_1} P_z(y)^{1-\varepsilon}dy \leqslant \Big(\int_{N_1} \big(P_z(y)^{1-\varepsilon}(1+\tau_{N_1}(y))^{\delta}\big)^{1/(1-\varepsilon)}dy\Big)^{1-\varepsilon}\Big(\int_{N_1}(1+\tau_{N_1}(y))^{-\delta/\varepsilon}dy\Big)^{\varepsilon}$$
$$\leqslant C_1^{\varepsilon}\big(1+\tau_{N_2}(z)\big)^{-(-1-\varepsilon)\sigma},$$

where σ is as in (3.17).

Proof of Theorem (3.1). We rewrite for kernels P_z the proof of Proposition 5.1 of [14] applying (3.27). Let

$$E_m^z = \{y\in N_1\colon\ P_z(y) > 2^{-m}\} \quad \text{for } m = 1, 2, \ldots$$

By Lemma (3.6) there are constants C_1, $C_2 > 0$ such that for all $z\in N_2$ the coordinates $|y_i|$ (see (2.14)) of a point $y\in E_m^z$ are no larger than $C_1\cdot 2^{C_2 m}$. Hence by Lemma (3.8) we can choose $\gamma > 0$ such that for all $z\in N_2$

$$\|y_1-y_2\| > 2^{-\gamma m}\sqrt{\chi}$$

provided $y_1\in E_m^z$, $y_2\notin E_{m+1}^z$. Now we divide N_1 into a lattice of cubes of side $2^{-\gamma m}$ defined in terms of the coordinates $y_1,\ldots,y_\chi$, $\chi = \dim N_1$. Let $Q_{m,j}^z$, $j = 1,\ldots,j_{m,z}$ be those cubes of this lattice which intersect E_m^z. Since $Q_{m,j}^z \subset E_{m+1}$, we have

$$j_{m,z} \leqslant 2^{\gamma m\chi}|E_{m+1}^z|$$

and, by (3.27),

$$j_{m,z} \leqslant C\cdot 2^{\gamma m\chi}\big(1+\tau_{N_2}(z)\big)^{-\alpha}\cdot 2^{(1-\varepsilon)m},$$

where α, ε and C are as in Lemma (3.26). Let

$$M_{m,j}^z f(x_1) = \sup_{h\in Z^\chi}\int_{Q_{m,j}^z} f(x_1\delta_h(y))dy;$$

then for a constant C_3 we have

$$M_z f(x_1) \leqslant C_3\sum_{m=1}^{\infty} 2^{-m}\sum_{j=1}^{j_{m,z}} M_{m,j}^z f(x_1).$$

Sjögren has proved in [14] that

$$\|M_{m,j}^z f\|_{L^p(N_1)} \leqslant C_4 m^{\chi/p}\cdot 2^{-\gamma m\chi}\|f\|_{L^p(N_1)},$$

where C_4 is a constant independent of z, m, j. Therefore

$$\|M_z f\|_{L^p(N_1)} \leqslant C_5\sum_{m=1}^{\infty} 2^{-m} m^{\chi/p} j_m\cdot 2^{-\gamma m\chi}\|f\|_{L^p(N_1)}$$
$$\leqslant C_6\big(1+\tau_{N_2}(z)\big)^{-\alpha}\|f\|_{L^p(N_1)},$$

which completes the proof.

REFERENCES

[1] R. Azencott and P. Cartier, *Martin boundaries of random walks on locally compact groups*, Proc. VI Berkeley Symposium on Math. Stat. and Prob., 1973, pp. 87–129.

[2] J. Birgé et A. Raugi, *Fonctions harmoniques sur les groupes moyennables*, C. R. Acad. Sci. Paris Sér. I Math. 278 (1974), pp. 1287–1289.

[3] J. M. Bony, *Principe du maximum inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) 19 (1969), pp. 277–304.

[4] E. Damek, *Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups*, Studia Math. 89 (1988), pp. 169–196.

[5] J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Inst. Hautes Études Sci. Publ. Math. 6 (1960), pp. 305–317.

[6] Y. Guivarc'h, *Une loi des grands nombres pour les groupes de Lie*, Séminaires de l'Université de Rennes, 1976.

[7] — *Quelques propriétés asymptotiques des produits de matrices aléatoires* in: École d'Été de Saint-Flour 1978, Lecture Notes in Math. 774, Springer, Berlin–Heidelberg–New York 1980.

[8] A. Hulanicki, *Subalgebra of $L_1(G)$ associated with laplacian on a Lie group*, Colloq. Math 31 (1974), pp. 259–287.

[9] — *A class of convolution semi-groups of measures on a Lie group*, pp. 82–101 in: *Probability Theory on Vector Spaces* (Proceedings, Błażejewko, Poland 1979), Lecture Notes in Math. 828, Springer, Berlin–Heidelberg–New York 1980.

[10] A. Korányi, *Poisson integrals and boundary components of symmetric spaces*, Invent. Math. 34 (1976), pp. 19–35.

[11] — *Compactifications of symmetric spaces and harmonic functions*, pp. 341–366 in: P. Eymar et al. (eds.), *Analyse harmonique sur les groupes de Lie II*, Lecture Notes in Math. 739, Springer, Berlin–Heidelberg–New York 1979.

[12] L. A. Lindahl, *Fatou's theorem for symmetric spaces*, Ark. Math. 10 (1972), pp. 33–47.

[13] A. Raugi, *Fonctions harmoniques sur les groupes localement compact à base dénombrable*, Bull. Soc. Math. France 54 (1977), pp. 5–118.

[14] P. Sjögren, *Admissible convergence of Poisson integrals in symmetric spaces*, Ann. Math. 124 (1986), pp. 313–335.

[15] E. M. Stein, *Boundary behavior of harmonic functions on symmetric spaces: maximal estimates for Poisson integrals*, Invent. Math. 74 (1983), pp. 63–83.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

Reçu par la Rédaction le 28.9.1988