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THE EXISTENCE OF $P(\alpha)$-POINTS OF N^* FOR $\aleph_0 < \alpha < \mathfrak{c}$

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In the presence of cardinals between $\aleph_0$ and $\mathfrak{c} = 2^{\aleph_0}$ there arises a possibility to differentiate P-points of $N^* = \beta N \setminus N$ by calling a point of N^* a *$P(\alpha)$-point* if it lies in the interior of the intersection of less than α its neighbourhoods; $P(\aleph_1)$-points are nothing else but usual P-points, and since $\mathfrak{c}$ is the greatest possible value for α, $P(\mathfrak{c})$-points will be called *absolute points* of N^*. In paper [6], by Kucia and the author, it was shown that in ZFC+MA (Martin's Axiom) there exist $2^{\mathfrak{c}}$ absolute points of N^*, which extended an earlier result of Blass [1] on the existence of $2^{\mathfrak{c}}$ P-points of N^* in ZFC+MA, as well as that of Rudin [9] that under CH (the Continuum Hypothesis, distinctions between P-points are unessential) there exist $2^{\mathfrak{c}}$ P-points of N^*. The problem arises whether for any α, $\aleph_0 < \alpha < \mathfrak{c}$, in ZFC+MA there exist $P(\alpha)$-points of N^* which are neither absolute nor even $P(\alpha^+)$-points. The main purpose of this paper is to give an affirmative answer to that problem in the only possible case of regular α, as the existence of $P(\alpha)$-points which are not $P(\alpha^+)$-points implies the regularity of α. Solomon [10] obtained a partial result that there exist $P(\aleph_1)$-points of N^* which are not $P(\aleph_2)$-points. Our result is obtained under assertion (S), weaker than MA if added to ZFC, which says, roughly speaking, that there exist no (α, β)-gaps (in a broad sense defined below) for any α and β between $\aleph_0$ and $\mathfrak{c}$. To get $P(\aleph_1)$-points which are not $P(\aleph_2)$-points (as in the Solomon result) it is sufficient to assume the non-existence of $(\aleph_0, \aleph_1)$-gaps on N in the usual sense (cf. Rothberger [8]).

The existence of P-points of N^* which are not absolute, under MA+ +non-CH if absolute points exist, shows that the subspace of N^* consisting of P-points is not homogeneous, in contrast to the situation from Rudin's paper [9] where CH is assumed.

1. Preliminaries. The following assertion (S), due to Martin and Solovay [7], is a known consequence of Martin's Axiom.

(S) *If R and P are families of non-empty closed-open subsets of N^* such that* $\mathrm{card}(R \cup P) < \mathfrak{c}$ *and if for each $A_1, \ldots, A_k \in R$, $k \in N$ (N is the* *set of positive integers), and for each $B \in P$, there is*

$$B \setminus (A_1 \cup \ldots \cup A_k) \neq \emptyset,$$

then there exists a closed-open subset C of N^* such that $C \cap A = \emptyset$ for each $A \in R$ and $C \cap B \neq \emptyset$ for each $B \in P$.

By a *gap* in a broad sense we mean the non-existence of a closed-open set C as in (S).

Assertion (S) is analogous to assertion Q of the non-existence of $(\aleph_0, \alpha)$-gaps on N for $\alpha < \mathfrak{c}$. The analogy fails if we substitute higher ordinals for $\aleph_0$. Namely, Hausdorff has shown [4], using merely the axiom of choice, that there exist $(\aleph_1, \aleph_1)$-gaps on N. Assertion Q was investigated by Rothberger [8] who proved that Q follows from an assertion, known presently as $P(\mathfrak{c})$, that each non-empty intersection of less than $\mathfrak{c}$ closed-open subsets of N^* has a non-empty interior; $P(\mathfrak{c})$ is a consequence of (S) (Booth [2]).

2. Some properties of towers. A *tower* is a transfinite sequence $\{T_\beta\colon \beta < \alpha\}$ of closed-open subsets of N^* such that $T_0 = N^*$, and $\gamma < \beta$ implies $T_\beta \subsetneqq T_\gamma$ (the terminology is taken from [3]). The symbol T will be used both for a tower and for the intersection of its members; the symbol $\mathrm{Bd}T$ will be used for the *boundary of tower* T, i.e. $\mathrm{Bd}T = T \setminus \mathrm{Int}T$. We assume that cardinals are initial ordinals (ordinals are usually denoted by small Greek letters). The *layers of tower* $T = \{T_\beta\colon \beta < \alpha\}$ are the sets

$$V_{\beta+1} = T_\beta \setminus T_{\beta+1} \quad \text{or} \quad V_\beta = \mathrm{Int} \bigcap \{T_\gamma\colon \gamma < \beta\} \setminus T_\beta$$

if β is a limit ordinal. The layers are open and disjoint. Moreover, the family of all layers of a tower T covers densely $N^* \setminus T$.

LEMMA 1 ($P(\mathfrak{c})$). *Let α be a regular cardinal. Let $T = \{T_\beta\colon \beta < \alpha\}$ be a tower and let $\mathcal{S}$ be a family of closed-open subsets of N^* such that $\mathrm{Bd}T \cap \bigcap \mathcal{S} \neq \emptyset$ and $\mathrm{card}\,\mathcal{S} < \alpha$. Then the set of all ξ, $\xi < \alpha$, for which the layer V_ξ intersects $\mathrm{Int} \bigcap \mathcal{S}$ is cofinal with α.*

Proof. Let β, $\beta < \alpha$, be fixed. Without loss of generality we may assume that $\mathcal{S}$ is closed under finite intersections. For any given A from $\mathcal{S}$ there exists a ξ_A, $\xi_A < \alpha$, such that

$$(T_\beta \setminus T_{\xi_A}) \cap A \neq \emptyset$$

(for if $(T_\beta \setminus T_\xi) \cap A = \emptyset$ for each $\xi < \alpha$, then $(T_\beta \setminus T) \cap A = \emptyset$, whence $T_\beta \cap A \subset T$, a contradiction with $\mathrm{Bd}T \cap \bigcap \mathcal{S} \neq \emptyset$). Since $\mathrm{card}\,\mathcal{S} < \alpha$ and α is regular, there exists a ξ, $\xi < \alpha$, such that $\xi_A < \xi$ for each $A \in \mathcal{S}$. For that ξ the sets $A \cap (T_\beta \setminus T_\xi)$ are non-empty and form a centered family of closed-open subsets of $T_\beta \setminus T_\xi$. By $P(\mathfrak{c})$, their intersection contains a non-empty open set W. In particular, $W \subset \mathrm{Int} \bigcap \mathcal{S}$. The family of all layers of T covers densely $N^* \setminus T$ and W is disjoint with T. Thus W meets a layer V_γ of T. We have

$$V_\gamma \cap \operatorname{Int} \bigcap \mathscr{S} \neq \emptyset.$$

It remains to prove that $\gamma \geqslant \beta$. In fact, $W \subset T_\beta \setminus T_\xi$, whence $V_\gamma \subset T_\beta \setminus T_\xi$ and thus $\gamma \geqslant \beta$.

LEMMA 2 (S). *Let α be a regular cardinal. Let $T = \{T_\beta\colon \beta < \alpha\}$ be a tower and let $\mathscr{S}$ be a family of closed-open subsets of N^* such that* $\operatorname{card}\mathscr{S} < \alpha < \mathfrak{c}$. *If $\bigcap \mathscr{S}$ intersects* $\operatorname{Bd}T$, *then* $\operatorname{Int}\bigcap\mathscr{S}$ *intersects* $\operatorname{Bd}T$.

Proof. Let B be the set of all ξ for which the layer V_ξ of T intersects $\operatorname{Int}\bigcap\mathscr{S}$. For each $\xi \in B$ take a non-empty closed-open set W_ξ such that

$$W_\xi \subset V_\xi \cap \operatorname{Int}\bigcap\mathscr{S}.$$

By Lemma 1, the set B is cofinal with α and, therefore, $\operatorname{cl}W \cap \operatorname{Bd}T \neq \emptyset$ for an arbitrary $W \subset N^*$ such that $W \cap W_\xi \neq \emptyset$ for each $\xi \in B$. Applying (S) to families $\{N^* \setminus A\colon A \in \mathscr{S}\}$ and $\{W_\xi\colon \xi \in B\}$, we get a closed-open subset G of N^* such that $G \cap (N^* \setminus A) = \emptyset$ for $A \in \mathscr{S}$ and $G \cap W_\xi \neq \emptyset$, $\xi \in B$. Hence

$$G \subset \bigcap\mathscr{S} \quad \text{and} \quad G \cap \operatorname{Bd}T \neq \emptyset.$$

Thus $\operatorname{Int}\bigcap\mathscr{S}$ intersects $\operatorname{Bd}T$.

LEMMA 3 (S). *Let α be a regular cardinal. Let $T = \{T_\beta\colon \beta < \alpha\}$ be a tower and let $\mathscr{S}$ be a family of closed-open subsets of N^* such that* $\operatorname{card}\mathscr{S} < \alpha < \mathfrak{c}$. *Let $\mathscr{R}$ be a family of closed-open subsets of N^* such that* $\operatorname{card}\mathscr{R} < \mathfrak{c}$. *If $\bigcap\mathscr{S}$ intersects* $\operatorname{Bd}T \cap \bigcap\mathscr{R}$, *then* $\operatorname{Int}\bigcap\mathscr{S}$ *intersects* $\operatorname{Bd}T \cap \bigcap\mathscr{R}$.

Proof. Without loss of generality we may assume that the family $\mathscr{R}$ is closed under finite intersections. Now, in virtue of Lemma 2, for each $R \in \mathscr{R}$ we have

$$\operatorname{Bd}T \cap R \cap \operatorname{Int}\bigcap\mathscr{S} \neq \emptyset.$$

Hence for each $R \in \mathscr{R}$ there exists a closed-open subset W_R of N^* such that

$$W_R \subset R \cap \operatorname{Int}\bigcap\mathscr{S} \quad \text{and} \quad W_R \cap \operatorname{Bd}T \neq \emptyset.$$

For $R \in \mathscr{R}$, denote by Q_R the family consisting of all non-empty sets $W_R \cap (T_\gamma \setminus T_\beta)$, where γ and β run over all ordinals less than α. Clearly, each Q_R consists of at most α closed-open subsets of N^* and, therefore, the family

$$Q = \bigcup\{Q_R\colon R \in \mathscr{R}\}$$

has the cardinality not greater than $\alpha \cdot \operatorname{card}\mathscr{R} < \mathfrak{c}$. Now, applying (S) to families $\{N^* \setminus A\colon A \in \mathscr{S}\}$ and Q, we get a closed-open subset G of N^* such that $G \cap (N^* \setminus A) = \emptyset$ for $A \in \mathscr{S}$ and $G \cap P \neq \emptyset$, $P \in Q$. Since $G \subset \operatorname{Int}\bigcap\mathscr{S}$, it suffices to show, in order to prove that $\operatorname{Int}\bigcap\mathscr{S}$ intersects $\operatorname{Bd}T \cap \bigcap\mathscr{R}$, that G intersects $\operatorname{Bd}T \cap \bigcap\mathscr{R}$. For this purpose, in virtue of the compactness of N^*, it suffices to show that the family

$$\{\mathrm{Bd}T\cap G\cap R\colon R\in\mathscr{R}\}$$

is centered. But to prove that, it suffices only to show that

$$\mathrm{Bd}T\cap G\cap R\neq\emptyset \quad \text{for } R\in\mathscr{R},$$

since $\mathscr{R}$ is closed under finite intersections. To do this it suffices to show that $\mathrm{Bd}T\cap G\cap W_R\neq\emptyset$, since $W_R\subset R$ for $R\in\mathscr{R}$. For this purpose assume, on the contrary, that $\mathrm{Bd}T\cap G\cap W_R=\emptyset$ for some $R\in\mathscr{R}$. By the compactness of N^*, there exists an ordinal γ, $\gamma<a$, such that

$$(T_\gamma\setminus\mathrm{Int}T)\cap G\cap W_R=\emptyset.$$

However, since $W_R\cap\mathrm{Bd}T\neq\emptyset$, there exists an ordinal β, $\beta<a$, such that $(T_\gamma\setminus T_\beta)\cap W_R\neq\emptyset$. Hence $(T_\gamma\setminus T_\beta)\cap W_R\in Q$ and, therefore,

$$(T_\gamma\setminus T_\beta)\cap W_R\cap G\neq\emptyset,$$

a contradiction, since $T_\gamma\setminus T_\beta\subset T_\gamma\setminus\mathrm{Int}\,T$.

3. Existence theorems.

LEMMA 4 (($P\mathfrak{c}$)). *If a is a regular cardinal such that $\aleph_0\leqslant a\leqslant\mathfrak{c}$, then there exists a tower $\{T_\beta\colon \beta<a\}$.*

A simple proof is omitted.

THEOREM (S). *If a is a regular cardinal such that $\aleph_0<a<\mathfrak{c}$, then there exists a $P(a)$-point of N^* which is not a $P(a^+)$-point.*

Proof. Since (S) implies $2^\gamma=\mathfrak{c}$ for $\gamma<\mathfrak{c}$ (Booth [3]) all families consisting of less than a closed-open subsets of N^* can be well ordered in the type $\mathfrak{c}$. Let $\{\mathscr{S}_\gamma\colon \gamma<\mathfrak{c}\}$ be a well ordering of those families. Let $T=\{T_\beta\colon \beta<a\}$ be a tower whose existence follows from Lemma 4. Now, for each γ, $\gamma<\mathfrak{c}$, we define a closed-open subset U_γ of N^* such that

1. if $U_\gamma\cap\bigcap\mathscr{S}_\gamma\neq\emptyset$, then $U_\gamma\subset\mathrm{Int}\bigcap\mathscr{S}_\gamma$;

2. $\{U_\gamma\cap\mathrm{Bd}T\colon \gamma\leqslant\beta\}$ is a centered family for each β, $\beta<\mathfrak{c}$.

In order to define U_0 consider the intersection of $\mathrm{Bd}\,T$ with $\bigcap\mathscr{S}_0$. If the intersection is empty, then we put a closed-open subset U_0 of N^*, disjoint with $\bigcap\mathscr{S}_0$, such that $U_0\cap\mathrm{Bd}T\neq\emptyset$. If the intersection is not empty, then, by Lemma 2,

$$\mathrm{Int}\bigcap\mathscr{S}_0\cap\mathrm{Bd}\,T\neq\emptyset$$

and, therefore, there exists a closed-open subset W of N^* such that

$$W\subset\mathrm{Int}\bigcap\mathscr{S}_0 \quad\text{and}\quad W\cap\mathrm{Bd}\,T\neq\emptyset.$$

Thus we put $U_0=W$.

Suppose that we have defined U_γ for $\gamma < \beta$, where $\beta < \mathfrak{c}$. By condition 2, the family $\{U_\gamma \cap \mathrm{Bd}\, T\colon \gamma < \beta\}$ is centered and consists of closed subsets of N^*. By the compactness of N^*,

$$K_\beta = \mathrm{Bd}\, T \cap \bigcap \{U_\gamma\colon \gamma < \beta\} \neq \emptyset.$$

In order to define U_β consider the intersection of K_β with $\bigcap \mathscr{S}_\beta$. If the intersection is empty, then we put U_β to be an arbitrary closed-open subset of N^* which is disjoint with $\bigcap \mathscr{S}_\beta$ and which intersects K_β. If the intersection is not empty, then, by Lemma 3,

$$K_\beta \cap \mathrm{Int} \bigcap \mathscr{S}_\beta \neq \emptyset$$

and, therefore, there exists a closed-open subset W of N^* such that

$$W \subset \mathrm{Int} \bigcap \mathscr{S}_\beta \quad \text{and} \quad W \cap K_\beta \neq \emptyset.$$

Then we put $U_\beta = W$.

It is clear that, in both cases, U_β satisfies, with defined previously U_β, $\gamma < \beta$, both conditions 1 and 2.

Now, by condition 2, the set $\mathrm{Bd}\, T \cap \bigcap \{U_\gamma\colon \gamma < \mathfrak{c}\}$ is not empty. We show that it consists of $P(a)$-points which are not $P(a^+)$-points. Let p be one of them. Since p lies in the boundary of tower T, p is not a $P(a^+)$-point. In order to show that p is a $P(a)$-point, let $\mathscr{R}$ be a family of less than a neighbourhoods of p. There exists $\mathscr{S}_\beta$ with $\beta < \mathfrak{c}$ such that $\bigcap \mathscr{S}_\beta \subset \bigcap \mathscr{R}$ and $p \in \bigcap \mathscr{S}_\beta$. Since

$$p \in \mathrm{Bd}\, T \cap \bigcap \{U_\gamma\colon \gamma < \mathfrak{c}\},$$

$p \in U_\beta$ and, therefore, $U_\beta \cap \bigcap \mathscr{S}_\beta \neq \emptyset$. Thus, by condition 1, $U_\beta \subset \mathrm{Int} \bigcap \mathscr{S}_\beta$. Hence U_β is a required neighbourhood of p contained in $\bigcap \mathscr{R}$.

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