COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 1

A CONVOLUTION PROPERTY OF THE CANTOR-LEBESGUE MEASURE

BY

DANIEL M. OBERLIN (TALLAHASSEE, FLORIDA)

Let T be the circle group R/Z and, for $1 \leqslant p < \infty$, let L^p be the usual Lebesgue space formed with respect to normalized Lebesgue measure m on T. It is well known that every complex Borel measure μ on T acts as a convolution operator on any L^p-space: $\mu * L^p \subseteq L^p$. More interesting is the fact that there are probability measures μ on T which are singular with respect to m and yet have the property that $\mu * L^p \subseteq L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$ and all $p \in (1, \infty)$. For examples of such μ obtained using Riesz products see p. 393 in [1]. For another example and a discussion of this phenomenon see p. 120-122 in [2]. The purpose of this note* is to prove the following

THEOREM. *Let λ be the Cantor-Lebesgue measure on T. For each $p \in (1, \infty)$ there is an $\varepsilon > 0$ such that $\|\lambda * f\|_{L^{p+\varepsilon}} \leqslant \|f\|_{L^p}$ for all $f \in L^p$.*

This theorem is a consequence of the following two lemmas:

LEMMA 1. *Suppose the inequality*

$$\left\{\frac{1}{3}\left[\left(\frac{a+b}{2}\right)^q + \left(\frac{b+c}{2}\right)^q + \left(\frac{a+c}{2}\right)^q\right]\right\}^{1/q} \leqslant \left(\frac{a^p+b^p+c^p}{3}\right)^{1/p} \tag{1}$$

holds for all positive numbers a, b, c. Then $\|\lambda * f\|_{L^q} \leqslant \|f\|_{L^p}$ for all $f \in L^p$.

LEMMA 2. *Inequality* (1) *is valid for $q = 2$ and $p = 2/(1+3^{-1/2}) \approx 1.2679$.*

For $2/(1+3^{-1/2}) \leqslant p < 2$, the Theorem is a direct consequence of the lemmas. For other values of $p < 2$, our result follows from the Riesz-Thorin theorem and the fact that $\|\lambda * f\|_{L^1} \leqslant \|f\|_{L^1}$ ($f \in L^1$). Duality and another application of complex interpolation take care of the case $2 \leqslant p < \infty$. Thus it is enough to prove the lemmas.

---

* Partially supported by NSF Grant MCS-7827602.

Proof of Lemma 1. For $N = 1, 2, \ldots$, let G_N be the cyclic group of 3^N elements realized as the set $\{0, 1, \ldots, 3^N-1\}$ with addition modulo 3^N, and let $L^p(G_N)$ be the Lebesgue space formed with respect to normalized counting measure on G_N. The norm in $L^p(G_N)$ will be denoted by $\|\cdot\|_{p,N}$. Let μ_N be the probability measure uniformly distributed over the set

$$S_N = \Big\{\sum_{j=0}^{N-1} \varepsilon_j \cdot 3^j \colon \varepsilon_j = 0, 2\Big\}.$$

We will show that if (1) holds for p and q, then

$$(2) \qquad \|\mu_N * f\|_{q,N} \leqslant \|f\|_{p,N}, \quad f \in L^p(G_N),\ N = 1, 2, \ldots$$

If we take the interval $[0, 1)$ as a model for T, then the Cantor-Lebesgue measure λ is the limit (in an appropriate sense) of the sequence of measures $\{\lambda_N\}_{N=1}^{\infty}$, where λ_N is the probability measure uniformly distributed over the set

$$\Big\{\sum_{j=0}^{N-1} \varepsilon_j \cdot 3^{j-N} \colon \varepsilon_j = 0, 2\Big\}.$$

Thus the conclusion of Lemma 1 will follow from (2) and an elementary limit argument. We will establish (2) by induction on N.

For $N = 1$, inequality (2) is a direct consequence of (1). So suppose that (2) is valid with N replaced by $N-1$ and let f be a function on G_N.

For $j = 0, 1, 2$, let $E_j = \{n \in G_N \colon n \equiv j \pmod 3\}$ and let

$$f_j(n) = \begin{cases} f(n) & \text{if } n \in E_j, \\ 0 & \text{if } n \notin E_j. \end{cases}$$

For $j = 0, 2$, let μ_N^j be the probability measure uniformly distributed over $S_N \cap E_j$. Thus $\mu_N = (\mu_N^0 + \mu_N^2)/2$. Now

$$(3) \quad \|\mu_N * f\|_{q,N}$$
$$= \Big\{\frac{1}{3^N}\Big[\sum_{n\in E_0}\Big(\frac{\mu_N^0 * f_0(n) + \mu_N^2 * f_1(n)}{2}\Big)^q + \sum_{n\in E_1}\Big(\frac{\mu_N^0 * f_1(n) + \mu_N^2 * f_2(n)}{2}\Big)^q +$$
$$+ \sum_{n\in E_2}\Big(\frac{\mu_N^0 * f_2(n) + \mu_N^2 * f_0(n)}{2}\Big)^q\Big]\Big\}^{1/q}$$
$$= \Big\{\frac{1}{3}\Big[\Big\|\mu_{N-1} * \frac{\tilde{f}_0 + \tilde{f}_1}{2}\Big\|_{q,N-1}^q + \Big\|\mu_{N-1} * \frac{\tilde{\tilde{f}}_1 + \tilde{f}_2}{2}\Big\|_{q,N-1}^q +$$
$$+ \Big\|\mu_{N-1} * \frac{\tilde{\tilde{f}}_2 + \tilde{\tilde{f}}_0}{2}\Big\|_{q,N-1}^q\Big]\Big\}^{1/q},$$

where $\tilde{f}_j$, $\tilde{\tilde{f}}_j$ are functions on G_{N-1} such that

$$\|\tilde{f}_j\|_{p,N-1}^p = \|\tilde{\tilde{f}}_j\|_{p,N-1}^p = 3\|f_j\|_{p,N}^p. \tag{4}$$

By way of example, we elaborate on the equality

$$\frac{1}{3^N}\sum_{n\in E_1}\left(\frac{\mu_N^0 * f_1(n) + \mu_N^2 * f_2(n)}{2}\right)^q = \frac{1}{3}\left\|\mu_{N-1} * \frac{\tilde{\tilde{f}}_1 + \tilde{f}_2}{2}\right\|_{q,N-1}^q.$$

Since $E_1 = E_0 + 1$ and $\mu_N^2(j) = \mu_N^0(j-2)$, the LHS of the above is

$$\frac{1}{3^N}\sum_{n\in E_0}\left(\frac{\mu_N^0 * f_1(n+1) + \mu_N^0 * f_2(n-1)}{2}\right)^q.$$

Putting $\tilde{\tilde{f}}_1(n) = f_1(n+1)$ and $\tilde{f}_2(n) = f_2(n-1)$, we obtain

$$\frac{1}{3^N}\sum_{n\in E_0}\left(\mu_N^0 * \frac{\tilde{\tilde{f}}_1 + \tilde{f}_2}{2}(n)\right)^q,$$

where $\tilde{\tilde{f}}_1$ and $\tilde{f}_2$ are supported on E_0. Now, identifying E_0 with G_{N-1} and μ_N^0 with μ_{N-1}, we get

$$\frac{1}{3}\left\|\mu_{N-1} * \frac{\tilde{\tilde{f}}_1 + \tilde{f}_2}{2}\right\|_{q,N-1}^q.$$

By (2) (with $N-1$ instead of N) and the triangle inequality, the last term of (3) is not greater than

$$\left\{\frac{1}{3}\left[\left(\frac{\|\tilde{f}_0\|_{p,N-1} + \|\tilde{f}_1\|_{p,N-1}}{2}\right)^q + \left(\frac{\|\tilde{\tilde{f}}_1\|_{p,N-1} + \|\tilde{f}_2\|_{p,N-1}}{2}\right)^q + \right.\right.$$

$$\left.\left. + \left(\frac{\|\tilde{\tilde{f}}_2\|_{p,N-1} + \|\tilde{\tilde{f}}_0\|_{p,N-1}}{2}\right)^q\right]\right\}^{1/q} \leqslant \left\{\frac{1}{3}[3\|f_0\|_{p,N}^p + 3\|f_1\|_{p,N}^p + 3\|f_2\|_{p,N}^p]\right\}^{1/p}$$

$$= \|f\|_{p,N}.$$

Here the inequality is a consequence of (1) and (4). Thus (2) is established and the lemma is proved.

Proof of Lemma 2. To study inequality (1) is essentially to examine the maxima of the quantity $(a+b)^q + (b+c)^q + (a+c)^q$ subject to the constraint $a^p + b^p + c^p = 1$. When $q = 2$, the method of Lagrange shows that if such a maximum occurs for a triple (a, b, c), then there is a constant

$\lambda < 0$ such that

$$a+\lambda a^{p-1} = b+\lambda b^{p-1} = c+\lambda c^{p-1}.$$

Since an equation $x+\lambda x^{p-1} = \text{const}$ $(\lambda < 0, 1 < p < 2)$ can have at most two solutions $x \geqslant 0$, at least two of the values a, b, c are equal. Thus, setting $a = b = t$ and $c = 1$, it suffices to show that for $p > 2/(1+3^{-1/2})$, the maximum of

$$f(t) = \frac{[(2t)^2+2(1+t)^2]^{1/2}}{(2t^p+1)^{1/p}}$$

for $t \geqslant 0$ occurs when $t = 1$.

Now $f'(t)$ has the same sign as $s(t) = -2t^p+3t-2t^{p-1}+1$. Since $s''(t) < 0$ for $t > (2-p)/p$ and since $s'(1) = 5-4p$, it follows that, for $p > 5/4$, $f(t)$ is decreasing for $t \geqslant 1$. A computation shows that $f(0) \leqslant f(1)$ if

$$p \geqslant 2\left(1+\frac{\log 2}{\log 3}\right)^{-1} \approx 1.2263.$$

Thus, for $p > 5/4$, it follows that if $f(t) > f(1)$ for any $t \geqslant 0$, then there exists t_1 with $0 < t_1 < 1$ and $s(t_1) = 0$, or

$$2t_1^{p-1} = 1+\frac{2t_1}{1+t_1}. \tag{5}$$

Let $y_1(t) = 2t^{p-1}$ and $y_2(t) = 1+2t/(1+t)$. If $5/4 < p_0 < 2$, it is easy to see that there exists $t_0 \in (0, 1)$ and $\varepsilon > 0$ such that, for $t_0 \leqslant t < 1$ and $p_0 \leqslant p \leqslant 2$, we have $y_1'(t)-y_2'(t) \geqslant \varepsilon$, so $y_2(t)-y_1(t) \geqslant (1-t)\varepsilon > 0$. Let S be the set of all $p \in [1.251, 2]$ for which there exists $t_1 \in [0, 1)$ such that (5) holds. It follows from the preceding remark that S is closed. Let p_1 be the greatest element of S. Then $p_1 < 2$. (If $S = \emptyset$, the lemma is proved.) We will show that

$$p_1 \leqslant 2/(1+3^{-1/2}), \tag{6}$$

which will complete the proof of the lemma.

For $p = p_1$, let $t_1 = \sup\{t \in [0, 1)\colon y_1(t) = y_2(t)\}$. Since $y_1(t) < y_2(t)$ for $t < 1$ and $|1-t|$ small, we have $t_1 < 1$. It then follows that $y_1'(t_1) = y_2'(t_1)$, and so

$$2(p_1-1)t_1^{p_1-1} = \frac{2t_1}{(1+t_1)^2}.$$

Since also

$$2t_1^{p_1-1} = y_1(t_1) = y_2(t_1) = 1+\frac{2t_1}{1+t_1},$$

we have

$$p_1 = 1 + \frac{2t_1}{1+4t_1+3t_1^2}.$$

But the function

$$g(t) = 1 + \frac{2t}{1+4t+3t^2}$$

satisfies $g(t) \leqslant 2/(1+3^{-1/2})$ for $0 \leqslant t \leqslant 1$. This establishes (6) and completes the proof of the lemma.

It would be interesting to determine the precise range of values p and q for which $\lambda * L^p \subseteq L^q$ (**P 1267**) and also to determine the range of values for which inequality (1) holds (**P 1268**). The only additional information we have concerning these problems is the following: if $\lambda * L^p \subseteq L^q$, then

$$\frac{1}{p} + \left(1 - \frac{\log 2}{\log 3}\right)\left(1 - \frac{1}{q}\right) \leqslant 1.$$

(Thus if $\lambda * L^p \subseteq L^2$, then $p \geqslant 2(1+\log 2/\log 3)^{-1} \approx 1.2263$.)

Added in proof. W. Beckner has shown that (1) holds with $q = 2$ precisely when $p > \log 4/\log 3 \approx 1.2619$.

REFERENCES

[1] A. Bonami, *Etude des coefficients de Fourier des fonctions de $L^p(G)$*, Annales de Institut Fourier (Grenoble) 20 (1970), p. 335-402.

[2] E. M. Stein, *Harmonic analysis on R^n*, p. 97-135 in: *Studies in harmonic analysis*, MAA Studies in Mathematics, Vol. 13, Mathematical Association of America, Washington, D. C., 1976.

DEPARTMENT OF MATHEMATICS
THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA

Reçu par la Rédaction le 20. 10. 1979;
en version modifiée le 20. 2. 1980