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# REMARKS ON FINITE TOPOLOGICAL SPACES

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Kuratowski [3] (see also [4], p. 48-49) has shown that at most 14 distinct sets can be constructed from a subset $A$ of a topological space $X$ by successive applications in any order of the closure operation and the complementation operation. Let us call sets thus constructed *relatives* of $A$. In a recent paper of Herda and Metzler [2], they proved that there exists a subset $A$ in a topological space of any cardinal $\geqslant 7$ from which exactly 14 relatives can be found. However, these 14 relatives cannot be co-existed in any space with less than 7 points.

The main result of the present paper is: If $A$ is a subset of an $n$-point space with $n \leqslant 7$, then $A$ has at most $2n$ relatives. For each $n$, the number of non-homeomorphic topologies under which the above construction is possible are also given in this paper. Moreover, as we know, not all subsets of $X$ would give rise to $2n$ relatives on itself ($n \leqslant 7$) no matter how good the topologies we have in hand. The set which has exactly $2n$ relatives will be called a $K_n$-set. Furthermore we shall show that our results hold only partially if the topological closure is replaced by a more generalized closure, namely the algebraic closure in abstract algebras.

Throughout this paper, we use $\bar{C}$ to denote the closure of the set $C$; $C'$ the complement of $C$. The number of elements of $C$ is denoted by $|C|$ and its interior is denoted by $\operatorname{int}(C)$.

Now let $A_1$ be a subset of the topological space $X$. We then denote $\bar{A}_1 = A_2$; $\bar{A}_1' = A_3, \ldots,$ and $A_1' = B_1$; $\bar{A}_1' = B_2; \ldots$ and so on. Under such construction we get distinct sets $A_1, \ldots, A_l$ and $B_1, \ldots, B_m$. They are the so called relatives of $A_1$. According to Kuratowski [3], $l \leqslant 7$, $m \leqslant 7$. Since each $A_{2k}$ is followed by the complement $A_{2k+1} = A_{2k}'$, $l$ must be odd. Similarly $m$ is odd.

**1. Finite topological spaces.** We shall study, in this section, the Kuratowski problem in the topological spaces with cardinal $\leqslant 7$. We are going to prove the following

THEOREM 1. *Let $X$ be a topological space and $A_1$ be a subset of $X$. If exactly $2n$ ($n \leqslant 7$) distinct sets can be constructed from $A_1$ by successive applications, in any order, of closure and complementation, then the number of elements of $X$ cannot be less than $n$.*

In order to prove this theorem, we start with some lemmas.

LEMMA 1. *If $l = 7$, then $|A_4| \geqslant 3$.*

Proof. Under our construction $A_3 = \overline{A}_1'$, $A_4 = A_1^{-'-}$, $A_7 = A_1^{-'-'-'}$. So clearly $A_3$ is an open set contained in $A_4$, and $A_7$ is the interior of $A_4$. Then we have $A_4 \supset A_7 \supset A_3$. Hence $|A_4| \geqslant 3$.

LEMMA 2. *If $l \geqslant 5$, $m \geqslant 5$, then $|A_5 \cap B_5| \geqslant 2$, $|A_5| \geqslant 3$ and $|B_5| \geqslant 3$.*

Proof. Observe that $A_5 = \operatorname{int}(\overline{A}_1) \supset \operatorname{int}(A_1) = B_3$, so $\overline{A}_5 \supset \overline{B}_3 = B_4$. Claim that $A_5 \cap B_5 \neq \emptyset$. For if $A_5 \cap B_5 = \emptyset$, then $A_5 \subset B_5' = B_4$ and so $\overline{A}_5 = A_6 \subset B_4$. Hence $A_6 = B_4$ which is a contradiction. Hence, by lemma 2 in [2], $|A_5 \cap B_5| \neq 1$, therefore $|A_5 \cap B_5| \geqslant 2$. As $A_5$ contains a non-void set $B_3$ which does not intersect $B_5$. So $|A_5| \geqslant 3$. By symmetry, $|B_5| \geqslant 3$.

LEMMA 3. *If $l \geqslant 5$ and $m \geqslant 3$, then $|A_5| \geqslant 2$.*

Proof. $B_3 \subset A_5$ implies $|A_5| \geqslant 1$. If $A_5 = \{a\}$, then $A_5' = A_4 = X \setminus \{a\}$ is a closed set and so $a$ cannot belong to $\overline{C} \setminus C$ for any set $C$. Thus $A_5 \subset A_2 = \overline{A}_1$ implies $a \in A_1$. Hence $a \notin B_1$ and so $a \in B_3$. Therefore it follows that $B_3 = A_5$ which is a contradiction. Hence $|A_5| \geqslant 2$.

LEMMA 4. *If the maximal number of relatives is attained, there is $|A_3| \leqslant 1$ and $|B_3| \leqslant 1$.*

Proof. It is easy to see that no proper subset of $A_3$ and $B_3$ appears in any $A_i$ or $B_i$.

We now are prepared to prove our Theorem 1. The proofs are different for different $n$.

If $2n = l+m = 12$, then there must be the case $l = 7$, $m = 5$ (or $l = 5$, $m = 7$). By lemma 1, $|A_4| \geqslant 3$; also by lemma 2, $|A_5| \geqslant 3$. As $A_5 = A_4'$, hence $|X| \geqslant 6$.

For $2n = l+m = 10$ there are two different possible cases: either $l = m = 5$ or $l = 7$, $m = 3$. In the former case, $|A_4'| = |A_5| \geqslant 3$ by lemma 2, and $A_3 \subset A_4$ implies $|A_4| \geqslant 2$. Thus $|X| \geqslant 5$. In the latter case, $|A_4| \geqslant 3$ by lemma 1, and $|A_5| \geqslant 2$ by lemma 3. Thus still $|X| \geqslant 5$.

For $2n = l+m = 8$ there are also two different possible cases: either $l = 7$, $m = 1$ or $l = 5$, $m = 3$. In the former case, $|A_4| \geqslant 3$ by lemma 1, and since $A_5 \neq \overline{A}_5 = A_6$, so $A_5 = A_4' \neq \emptyset$. Hence $|X| \geqslant 4$. In the latter case, $|A_5| \geqslant 2$ by lemma 3 and the inclusion $A_2 \supset A_5$ gives $|A_2| \geqslant 3$. As $A_2' = A_3 \neq \overline{A}_3 = A_4$, so $A_2' \neq \emptyset$. Thus $|X| \geqslant 4$.

The cases $n < 4$ are trivial, we omit the proofs.

For $n = 7$, it is proved by Herda and Metzler in [2].

**2. Number of non-homeomorphic topologies.** Throughout this section, we let $X$ be a finite space with cardinal $n$ and let $A_1$ be a potential $K_n$-set of $X$, that is $A_1$ may produce $2n$ relatives under some topologies. Such kind of topologies are said to be "good". Obviously, not all topologies given to $X$ are "good". In this section, we shall find out the number of non-homeomorphic "good" topologies for each $n$. Denote this number by $T(n)$; of course, the value of $T(n)$ depends on $n$. By the result of Herda and Metzler [2], we need only to consider the cases $n \leqslant 7$.

PROPOSITION 1. $T(2) = 2$.

Proof. Suppose $X = \{a, b\}$ and let $A_1 = \{a\}$. If $\{\bar{a}\} = \{a, b\}$, then we have $A_2 = \{a, b\}$, $A_3 = \emptyset$, $B_1 = \{b\}$. The set $\{b\}$ can be closed or not. So it gives two distinct topologies.

PROPOSITION 2. $T(3) = 3$.

Proof. Let $X = \{a, b, c\}$. In order to obtain six different sets from a subset of $X$ under our construction, there must be a set with a non-trivial closure. Let us put $\overline{\{a\}} = \{a, b\}$. We then get two distinct topologies by setting $\overline{\{b\}} = \{a, b\}$ and $\overline{\{c\}} = \{c\}$ or $\overline{\{c\}} = X$. Starting with $A_1 = \{a\}$, we get six relatives of $A_1$, so these two topologies are "good". A third topology can be obtained by taking $\{b\}$ closed and $\{\bar{c}\} = X$.

PROPOSITION 3. $T(4) = 5$.

Proof. Let $X = \{a, b, c, d\}$. There can be $l = 7$, $m = 1$ or $l = 5$, $m = 3$. In the first case, we can find two possible closed sets with cardinal 3, namely the set $A_4$ by lemma 1 and the set $A_2$ by lemma 4. Since $A_2$ is closed, so $\overline{\operatorname{int}(A_2)} \subsetneq A_2$, that is, $A_6 \subsetneq A_2$. As $|A_2| = 3$ and $\operatorname{int}(A_2) \neq \emptyset$, so $|A_6| = 2$. Now let us put $A_2 = \{a, b, d\}$, $A_4 = \{b, c, d\}$ and $A_6 = \{a, b\}$. Their intersections $\{b, d\}$, $\{b\}$ must therefore be closed. Considering $\{d\}$ to be closed or not, we obtain two distinct topologies. In the second case, we easily obtain two closed sets, namely $A_2 = \{a, b, d\}$ by lemma 4 and $A_4 = \{c, d\}$ by lemma 3. Their intersection $\{d\}$ is also closed. So we can only define $\overline{\{a\}} = \{a, b, d\}$ or $\{a, b\}$; $\overline{\{b\}} = \{b\}$ either $\{a, b\}$ either $\{a, b, d\}$ or $\{b, d\}$. This gives us three new non-homeomorphic topologies. Starting with the set $\{a, d\}$, we can verify that all these topologies are "good".

PROPOSITION 4. $T(5) = 7$.

Proof. Let $X = \{a, b, c, d, e\}$. There are two different possible cases: either $l = 7$, $m = 3$ or $l = m = 5$. In the former case, we easily obtain three closed sets, namely the set $A_2 = \{a, b, c, d\}$ by lemma 4, $A_4 = \{e, c, d\}$ by lemmas 1 and 3, and the set $A_6$ which is a closed proper subset of $A_2$. By lemma 3, $A_6 = \{a, b, d\}$. So their intersections $\{c, d\}$ and $\{d\}$ are closed. Since the space $X$ is finite, we can describe the topologies by taking the closure pointwise. Under our construction, the followings are bounded to be happened: $\overline{\{a\}} = \{a, b, d\}$, $\overline{\{d\}} = \{d\}$, $\overline{\{e\}} = \{c, d, e\}$.

For the sets $\{b\}$ and $\{c\}$ it is then only possible to define $\overline{\{b\}} = \{b\}$, $\{b, d\}$ or $\{a, b, d\}$ and $\overline{\{c\}} = \{c\}$ or $\{c, d\}$. In this way we obtain six topologies and five of them are non-homeomorphic. Starting with the set $\{a, c\}$. We can verify that all of the above topologies are "good". In the latter case, $l = m = 5$. We have $|A_5| = 3$, this is obtained by the fact that $A_5' = A_4 = \bar{A}_3$ implies $|A_5'| \geqslant 2$ and by lemma 2. Put $A_5 = \{a, b, c\}$. Also by lemma 4, we have $|A_3| = |B_3| = 1$. Put $A_3 = \{e\}$, $B_3 = \{a\}$. Then $A_4 = A_5' = \{d, e\}$, so it follows that $\overline{\{e\}} = \{d, e\}$. By lemma 2, $\{b, c\} \subset A_5 \cap B_5$, $|B_3| = 3$, and so $\overline{\{a\}} = B_5' = \{a, d\}$. It follows $\overline{\{d\}} = \{d\}$ since $\{d\}$ is the intersection of $\overline{\{a\}}$ and $\overline{\{e\}}$. Moreover, it is easy to see that $b \in A_1$, $c \in A_2 \setminus A_1$ or vice-versa. Then, by $\{b, c\} \subset A_2 \cap B_2$, we have $b \in \overline{\{c\}}$ and $c \in \overline{\{b\}}$, thus $\overline{\{b\}} = \overline{\{c\}}$. Now define $\overline{\{b\}} = \{b, c\}$ or $\{b, c, d\}$. These give another two "good" topologies.

PROPOSITION 5. $T(6) = 4$.

Proof. Let $X = \{a, b, c, d, e, f\}$. The only case is $l = 7$, $m = 5$. By lemma 4, we can put $A_2 = \{a, b, c, d, e\}$, and by lemmas 1 and 2, put $A_4 = \{d, e, f\}$, $A_5 = \{a, b, c\}$. Since $\bar{A}_5 = A_6 \neq A_5$, this gives $A_6 = \{a, b, c, d\}$. By lemma 2, we may assume that $\{b, c\} \subset B_5$. With this and by lemma 4, we have $B_3 = \{a\}$, $B_4 = \bar{B}_3 = \{a, d\}$. Therefore closures of $\{d\}$, $\{f\}$, and $\{a\}$ are fixed. Moreover, $\{b\}$ and $\{c\}$ has the same closure (see [3], lemma 2). For the rest of points, it is only possible for us to define $\overline{\{e\}} = \{e\}$ or $\{d, e\}$ and $\overline{\{b, c\}} = \{b, c\}$ or $\{b, c, d\}$. These give us 4 non--homeomorphic topologies. Starting with the set $A_1 = \{a, b, e\}$, we can verify that our topologies are "good".

PROPOSITION 6. $T(7) = 6$.

Proof. Let $X = \{a, b, c, d, e, f, g\}$. By lemma 4, we can put $A_2 = \{a, b, c, d, e, f\}$. As $\operatorname{int}(A_4) = A_7 \supset A_3$, so $|A_7| \geqslant 2$. By symmetry, $|B_7| \geqslant 2$. $B_7$ is an open set contained in $A_2$ and so $B_7 \subset A_5 = \operatorname{int}(A_2)$. But by lemma 3, $|A_5 \cap B_5| \geqslant 2$, so $|A_5| \geqslant 4$. Since $A_5 = A_4'$, by lemma 1 it follows that $|A_5| = 4$. By symmetry, $|B_5| = 4$ and hence $|B_4| = |B_5'| = 3$. Put $A_4 = \{e, f, g\}$, $A_5 = \{a, b, c, d\}$. The fact $\bar{A}_5 = A_6 = A_7'$ gives us $A_6 = \{a, b, c, d, e\}$ and $A_7 = \{f, g\}$. Now let us put $B_2 = \{b, c, d, e, f, g\}$, $B_4 = \{a, b, e\}$, $B_6 = \{c, d, e, f, g\}$. Therefore we obtain $\overline{\{g\}} = \{e, f, g\}$, $\overline{\{a\}} = \{a, b, e\}$, and $\overline{\{e\}} = \{e\}$ as $\{e\} = A_6 \cap A_4$. The sets $\{b, e\} = B_2 \cap B_4$ and $\{e, f\} = A_2 \cap A_4$ must be closed. Hence $\overline{\{b\}} = \{b\}$ or $\overline{\{b\}} = \{b, e\}$, and $\overline{\{f\}} = \{f\}$ or $\overline{\{f\}} = \{e, f\}$. By lemma 2 in [3], $\overline{\{c\}} = \overline{\{d\}}$. We can put either $\overline{\{c, d\}} = \{c, d\}$ or $\overline{\{c, d\}} = \{c, d, e\}$. In this way, we obtain six distinct topologies(¹). Starting with the set $A_1 = \{a, c, f\}$, we can verify that these topologies are "good".

Concluding these six propositions, we obtain immediately the following

(¹) The Theorem 2 given by Herda and Metzler in [2] is not true. They did not consider the possibility of inserting a number four into the set {1, 2, 6, 7}.

THEOREM 2. *Let $X$ be an $n$-point set with $n \leqslant 7$. If $A_1$ is a subset of $X$ which is a $K_n$-set under some "good" topology, then the number of non-homeomorphic "good" topologies $T(n)$ for each $n$ can be determined and is shown by the following table*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $T(n)$ | 1 | 2 | 3 | 5 | 7 | 4 | 6 |

**3. Algebraic closure.** In this section, we introduce the notions of abstract algebras and algebraic closure. We shall show that our results obtained in the preceding sections do not hold if the topological closure is replaced by the algebraic closure.

By an *abstract algebra* $(X; F)$ we mean a set $X$ and a family of fundamental operations consisting of $X$-valued functions of several variables running over $X$. If $X = \{a, b, \ldots\}$ and $F = \{\varphi, \psi, \ldots\}$, we shall sometimes write $(a, b, \ldots; \varphi, \psi, \ldots)$ or $(X; \varphi, \psi, \ldots)$ instead of $(X; F)$.

The $n$-ary operations $e_k^n(x_1, \ldots, x_n) = x_k$ $(k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ are called *trivial*. The smallest class which contains trivial operations and is closed under composition of these fundamental operations is called the *class* of algebraic operations. The values of constant algebraic operations are called algebraic constants. If $A$ is a non-void subset of $X$, then the smallest subalgebra containing $A$, that is, the smallest set which contains $A$ and is closed with respect to the algebraic operations, will be called *algebraic closure* of $A$ and denoted by $\hat{A}$. $\hat{\Phi}$ is the set of algebraic constants. The algebraic closure of $A$ satisfies the following three axioms:

(i) $A \subset \hat{A}$,

(ii) If $B \subset \hat{A}$, then $\hat{B} \subset \hat{A}$,

(iii) $\hat{\hat{A}} = \hat{A}$.

More about algebraic closure and abstract algebras the reader can find in [5]. The following is easy to prove.

THEOREM 3. *Let $(X; F)$ be an abstract algebra, $A$ a subset of $X$. Then at most* 14 *distinct sets can be constructed from $A$ by taking the algebraic closure and complementation successively in any order.*

Proof. Just need to observe that lemmas 3 and 4 remain valid for algebraic closure and observe that the inclusion $A_3 \subset A_7 \subset A_4$ is true. (See [5].)

We have seen in section 1 that the maximal number of relatives of a subset $A_1$ in an $n$-element topological space $X$ is $2n$ $(n \leqslant 7)$. Naturally, one would ask whether the same thing holds in abstract algebras. The answer is not. In fact, let $X$ be an $n$-element algebra, $A_1$ be a subalgebra of $X$ which gives the maximal number of relatives. Denote this maximal number by $V(n)$ for each $n$. We claim that

(i) $V(3) = 8$.

Proof. Let $(X; F) = (a, b, c; \varphi, \psi)$ where $\varphi(x) = b$, $\psi(a) = \psi(b) = b$, $\psi(c) = a$. Start with $\{a\}$. Observe that $\{\hat{a}\} = \{a, b\}$, $\{\hat{c}\} = X$, $\Phi = \{b\}$.

(ii) $V(4) \leqslant 10$.

Proof. If $l = 7$, $m \geqslant 5$, then $|A_5| \geqslant 2$ by lemma 3. $A_5 \subset A_6 \subset A_2$ implies $|A_2| \geqslant 4$. Thus if $|X| = 4$, then $A_2 = X$. This implies $A_4 = \hat{\Phi}$ and so $A_6 = \hat{A}_5 = \hat{A}_4' = X = A_2$. Contradiction.

Example 1. Let $(X; F) = (a, b, c; \varphi, \psi)$ where $\varphi(x) = b$, $\psi(d) = a$, $\psi(a) = d$, $\psi(b) = \psi(c) = b$. Start with $A_1 = \{a, c\}$, we obtain 10 distinct sets.

(iii) $V(5) \leqslant 12$.

Proof. If $l+m = 14$, then $|A_3| \geqslant 1$, $|B_3| \geqslant 1$ because $A_6 \neq A_2$. Hence $|B_4| = |\hat{B}_3| \geqslant 2$, $|A_2| = |A_3'| \leqslant 4$. $B_4 \subset A_6 \subset A_2$ implies $|B_4| = 2$ and $|B_5| = 3$. But $B_5 \subset B_6 \subset B_2$ implies $|B_2| \geqslant 5$ which is a contradiction.

Example 2. Let $(X; F) = (a, b, c, d, e; \varphi, \psi)$, where

$$\varphi(x) = \begin{cases} d \text{ if } x = e, \\ c \text{ if } x \neq e, \end{cases} \qquad \psi(x, y) = \begin{cases} b \text{ if } (x, y) = (a, d), \\ c \text{ if } (x, y) \neq (a, d). \end{cases}$$

Start with $A_1 = \{a, d\}$; we obtain 12 distinct sets.

Example 3. Let $(X; F) = (a, b, c, d, e, f; \varphi, \psi)$, where

$$\varphi(x) = \begin{cases} b \text{ if } x = a, \\ e \text{ if } x = f, \\ d \text{ if } x \notin \{a, f\}, \end{cases} \qquad \psi(x, y) = \begin{cases} c \text{ if } x = b \text{ and } y = f, \\ d \text{ if } (x, y) \neq (b, f). \end{cases}$$

Start with $A_1 = \{a, c, e\}$; we get 14 distinct sets.

Concluding the above results, we obtain

THEOREM 4. *Let $X$ be an $n$-element set. Then there exists an abstract algebra $(X; F)$ and a subset $A_1$ of $X$ such that, by taking algebraic closure and complemention operations to $A_1$, in any order, we obtain*

(1) $2^n$ *relatives of $A$ if $n \leqslant 3$,*
(2) $2(n+1)$ *relatives of $A$ if $3 \leqslant n \leqslant 6$,*
(3) 14 *relatives of $A$ if $n \geqslant 6$.*

The number of relatives of $A$ cannot be enlarged.

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