COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

REGULARITY OF BANACH LATTICE VALUED MARTINGALES

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In the present paper we consider natural analogues of classical martingale theorems in Banach lattices. We obtain also martingale characterization of order or geometric structure of the underlying Banach lattice.

1. Introduction. In the theory of classical real martingales there is a principal fact known as

BASIC SUBMARTINGALE THEOREM (cf. [4], p. 63). *Every integrable submartingale* $\{X_n, n \in N\}$ *satisfying the condition*

$$\sup_n \mathrm{E} X_n^+ < \infty$$

converges a.s. to an integrable random variable.

Now, a submartingale taking values in a Banach space equipped with a partial ordering can be defined in a natural way. In [7] one can find a try of a generalization of the theorem above in this direction. However, the above-mentioned result is not a copy of the real-case theorem. A submartingale, in general, is required to have some stronger properties to be convergent. On the other hand, this situation gives us a possibility of characterization of the underlying structure of a Banach space (cf. [6] and [7]).

Section 3 contains some results connected with the decomposition theorem; then some analogues of the Basic Submartingale Theorem are considered. These are stronger than that contained in [6], and also the class of Banach lattices, where the Basic Submartingale Theorem holds true, is described. In addition, we give some application to the cone of absolutely summing operators.

2. Notation. Throughout the paper $\mathscr{X}$ denotes a Banach lattice, i.e., a vector lattice equipped with the monotone ($|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$) and complete norm. We write, as usual,

$$x^+ = \sup(x, 0), \quad x^- = \sup(-x, 0), \quad |x| = \sup(x, -x).$$

For any elementary facts concerning vector lattices used in the text consult [3] and [5].

By $\mathscr{X}_+$ we denote the cone of non-negative elements from $\mathscr{X}$. A Banach lattice is said to be an *abstract* L-*space* (AL-*space*) if

$$\|x+y\| = \|x\|+\|y\| \quad \text{for all } x, y \in \mathscr{X}_+. \tag{1}$$

A Banach lattice is said to be a KB-*space* if no sublattice is isomorphic to c_0 (the Banach lattice of all real null sequences with the supremum norm and the coordinatewise ordering).

Now, let $(\Omega, \mathscr{F}, P)$ be a probability space and let $p \geqslant 1$. We denote by $L_p(\Omega, \mathscr{F}, P; \mathscr{X}) = L_p(\mathscr{X})$ a Banach lattice of all strongly measurable functions $X\colon \Omega \to \mathscr{X}$ such that

$$\|X\|_p = (\mathrm{E}\,\|X\|^p)^{1/p} < \infty.$$

Obviously, $X \leqslant Y$, $X, Y \in L_p(\mathscr{X})$, if $X \leqslant Y$ a.s. A sequence $\{X_n, n \in N\} \subset L_p(\mathscr{X})$ is said to be a *p-integrable submartingale* (if $p = 1$, briefly, submartingale) with respect to the increasing family of sub-σ-fields $\{\mathscr{F}_n\}$ if each X_n is $\mathscr{F}_n$-measurable and

$$\mathrm{E}(X_{n+1}|\mathscr{F}_n) \geqslant X_n \quad \text{for all } n \in N. \tag{2}$$

X_n is called a *supermartingale* if $\geqslant$ in (2) is replaced by $\leqslant$, and X_n is called a *martingale* if it is both a submartingale and a supermartingale. A martingale is called *regular* if it can be decomposed as a difference of positive martingales. From now on all Banach spaces considered are assumed to be separable.

3. Krickeberg decomposition. We start with an immediate generalization of the non-random situation:

LEMMA 1. *Let $\mathscr{X}$ be a* KB-*space. Let $\{A_n, n \in N\}$ be an increasing positive sequence of random vectors. If*

$$\sup_n \mathrm{E}\,\|A_n\|^p < \infty,$$

then there exists an $A \in L_p(\mathscr{X})$ such that $A_n \to A$ a.s. and in $L_p(\mathscr{X})$. Further, $A = \sup\{A_n\}$.

Proof. Let $\{A_n\} \subset L_p(\mathscr{X})$ satisfy the assumptions of the lemma. Then

$$\sup_n \|A_n(\omega)\| < \infty \quad \text{for almost all } \omega \in \Omega.$$

Since $\mathscr{X}$ is a KB-space, A_n converges a.s. to some random vector A (cf. Proposition 5.15 in [5]). By the Lebesgue theorem,

$$A \in L_p(\mathscr{X}) \quad \text{and} \quad A = \lim_n A_n \text{ in } L_p(\mathscr{X}).$$

Also, $A = \sup\{A_n\}$ (cf. Theorem 5.9 in [5]), which completes the proof of Lemma 1.

Now, a well-known Krickeberg decomposition theorem (cf. [4], p. 63) can easily be generalized into a Banach lattice.

PROPOSITION 1. *Let $\mathscr{X}$ be a* KB*-space. Let $\boldsymbol{X} = \{X_n\}$ be a submartingale satisfying the condition*

$$\sup_n \mathrm{E}\,\|X_n^+\| < \infty.$$

Then $\boldsymbol{X}$ can be decomposed in the form $\boldsymbol{X} = \boldsymbol{U} - \boldsymbol{V}$, where $\boldsymbol{U} = \{U_n\}$ is a positive martingale and $\boldsymbol{V} = \{V_n\}$ is a positive supermartingale.

Remark 1. If $\boldsymbol{X}$ is a martingale, then, clearly, $\boldsymbol{V}$ is also a martingale.

Proof. Since $\{X_k^+, k \in N\}$ is a positive submartingale, $\{Y_k^n, k \geqslant n\}$ is a positive increasing sequence, where

$$Y_k^n = \mathrm{E}(X_k^+ | \mathscr{F}_n) \quad \text{for } k \geqslant n.$$

Also

$$\sup_{k \geqslant n} \mathrm{E}\,\|Y_k^n\| < \infty \quad \text{for all } n \in N.$$

By Lemma 1 there exists a limit

$$U_n = \lim_k Y_k^n \quad \text{for all } n \in N, \tag{3}$$

and $\boldsymbol{U} = \{U_n\}$ is a martingale, since

$$\mathrm{E}(U_{n+1} | \mathscr{F}_n) = \mathrm{E}(\lim_k Y_k^{n+1} | \mathscr{F}_n) = \lim_k \mathrm{E}(Y_k^{n+1} | \mathscr{F}_n)$$

$$= \lim_k \mathrm{E}(X_k^+ | \mathscr{F}_n) = U_n.$$

Since $\mathrm{E}(X_k^+ | \mathscr{F}_n) \geqslant X_n^+$ for $k \geqslant n$, we have $\boldsymbol{U} \geqslant \boldsymbol{X}$, i.e. $U_n \geqslant X_n$ for all $n \in N$. Therefore $X_n = U_n - V_n$, where

$$V_n \overset{\mathrm{df}}{=} U_n - X_n \geqslant 0. \tag{3'}$$

This completes the proof of Proposition 1.

Remark 2. The decomposition obtained in Proposition 1 is the best in the following sense:

If $\boldsymbol{X} = \boldsymbol{U}' - \boldsymbol{V}'$, where $\boldsymbol{U}'$ is a positive martingale and $\boldsymbol{V}'$ is a positive supermartingale, then $\boldsymbol{U} \leqslant \boldsymbol{U}'$ and $\boldsymbol{V} \leqslant \boldsymbol{V}'$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are defined in (3) *and* (3'), *respectively.*

The above-mentioned and connected results can be expressed in terms of lattice theory. That is, let us denote by

$$\mathfrak{M}_p(\Omega, \mathscr{F}, P; \mathscr{X}) = \mathfrak{M}_p(\mathscr{X})$$

a vector space of $\mathscr{X}$-valued p-integrable martingales defined on $(\Omega, \mathscr{F}, P)$. The vector space $\mathfrak{M}_p(\mathscr{X})$ can be partially ordered by the canonical ordering of $[L_p(\mathscr{X})]^\infty$. Now we have

PROPOSITION 2. *Let $\mathscr{X}$ be a* KB*-space and let $M \in \mathfrak{M}_p(\mathscr{X})$. Consider the following statements*:

(i) $\sup \mathbf{E}\|M_n^+\|^p < \infty$;

(ii) *M is regular*;

(iii) *$M = \sup(M, -M)$ exists in $\mathfrak{M}_p(\mathscr{X})$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. It suffices to show (ii) $\Leftrightarrow$ (iii), since the remaining implications follow by the arguments similar to those given in Proposition 1.

So, first, let $M = U - V$, where $U, V \in [\mathfrak{M}_p(\mathscr{X})]_+$. Since

$$\mathbf{E}(M_k^+ | \mathscr{F}_n) \leqslant U_n \quad \text{for all } n \in N \text{ and } k \geqslant n,$$

by Remark 2 there exists

$$M^+ = \sup(M, 0) \quad \text{in } \mathfrak{M}_p(\mathscr{X}),$$

where

$$M^+ = \{\lim_k \mathbf{E}(M_k^+ | \mathscr{F}_n),\ n \in N\}.$$

Similarly,

$$M^- = \sup(-M, 0) = \{\lim_k \mathbf{E}(M_k^- | \mathscr{F}_n),\ n \in N\}$$

exists in $\mathfrak{M}_p(\mathscr{X})$, and so

$$M \overset{\text{df}}{=} M^+ + M^- = \sup(M, -M).$$

On the other hand, (iii) $\Rightarrow$ (ii) follows clearly since

$$M = |M| - (|M| - M),$$

so the proof is completed.

One may ask when (i) of Proposition 2 is equivalent to (ii) or (iii). The answer to this and other questions is contained in the next section.

4. Submartingale convergence theorems. We say that a Banach lattice satisfies the *Submartingale Boundedness* (*Convergence*) *Theorem* if any $\mathscr{X}$-valued submartingale satisfying the condition

$$\sup_n \mathbf{E}\|X_n^+\| < \infty$$

is bounded in $L_1(\mathscr{X})$ (converges a.s. to some $X \in L_1(\mathscr{X})$). We will write $\mathscr{X} \in$ SBT ($\mathscr{X} \in$ SCT).

PROPOSITION 3. *Let $\mathscr{X}$ be a* KB*-space. Then the following statements are equivalent*:

(i) $\mathscr{X} \in$ SBT ($\mathscr{X} \in$ SCT).

(ii) *Any positive supermartingale is bounded in $L_1(\mathscr{X})$ (converges a.s. to some $X \in L_1(\mathscr{X})$).*

(iii) *Any positive martingale is bounded in $L_1(\mathscr{X})$ (converges a.s. to some $X \in L_1(\mathscr{X})$).*

(iv) *Any martingale with*

$$\sup_n \mathbf{E}\,\|X_n^+\| < \infty$$

is bounded in $L_1(\mathscr{X})$ (*converges a.s. to some* $X \in L_1(\mathscr{X})$).

Proof. We prove Proposition 3 for L_1-boundedness since similar arguments can be used for the underlying convergence.

(i) $\Rightarrow$ (ii). Let $\{X_n\}$ be a positive supermartingale. Then $\{-X_n\}$ is a submartingale with $\sup \mathbf{E}\,\|(-X_n)^+\| < \infty$. Thus

$$\sup_n \mathbf{E}\,\|X_n\| < \infty.$$

(ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (iv) follows immediately from Proposition 1.

(iv) $\Rightarrow$ (i) holds by arguments similar to those used in the proof of Theorem 4.1 in [7].

Remark 3. In view of the Chaterji theorem it is easy to check that $\mathscr{X} \in \mathrm{SCT}$ iff $\mathscr{X} \in \mathrm{SBT}$ and $\mathscr{X}$ has the Radon-Nikodym property. Hence $\mathscr{X} \in \mathrm{SCT}$ iff $\mathscr{X}$ is isomorphic to a sublattice of l_1 (cf. Theorem 2 in [6]).

THEOREM 1. *The following statements are equivalent*:

(i) $\mathscr{X} \in \mathrm{SBT}$.

(ii) *For any sequence of positive independent real random variables* $\{f_n\}$ *and for any sequence of positive vectors* $\{x_n\}$ *we have* $\{x_n\} \subset \mathscr{X}_+$.

(iii) $\mathscr{X}$ *is isomorphic to an* AL*-space.*

LEMMA 2. *Let us put*

$$\mathscr{Z} = \left\{x = \{x_i\} \in \mathscr{X}^\infty\colon \|x\|_{\mathscr{Z}} \stackrel{\mathrm{df}}{=} \sup\left\{\left\|\sum_F |x_i|\right\|\colon F \subset N \text{ is finite}\right\} < \infty\right\}$$

and

$$\mathscr{Z}_p = \left\{x = \{x_i\} \in \mathscr{X}^\infty\colon \|x\|_{\mathscr{Z}_p} \stackrel{\mathrm{df}}{=} \sup\left\{\left\|\sum_F |f_i x_i|\right\|_p\colon F \subset N \text{ is finite}\right\} < \infty\right\},$$

where $\{f_i\}$ *is a sequence of positive real random variables with finite* p*-th moments. Then* $\mathscr{Z}$ *and* $\mathscr{Z}_p$ *are Banach lattices under the natural ordering.*

Proof. Evidently, $\mathscr{Z}$ and $\mathscr{Z}_p$ are vector lattices with monotone norms. The completeness of the defined norms follows from standard functional analysis arguments (cf. [2], Theorem 5.2).

LEMMA 3. *There exists a sequence of positive independent real random variables* $\{f_n^0\}$ *such that in an arbitrary Banach lattice the inequality*

$$\sum_{i=1}^n \|x_i\|^p \leqslant C\left\|\sum_{i=1}^n f_i^0 x_i\right\|_p^p \quad \text{for every } n \in N,\ x_1, \ldots, x_n \in \mathscr{X},$$

holds, where C *is some positive constant.*

Proof. Let $\{a_n\}$ be a sequence of reals such that

$$0 < a_n < 1 \text{ for all } n \in N \quad \text{and} \quad \prod_{i=1}^{\infty}(1-a_i) = a > 0.$$

Let $\{S_n\}$ be a sequence of independent events with $P(S_n) = a_n$. Put

$$f_n^0 = a_n^{-1} I(S_n),$$

where $I(\cdot)$ is the indicator function. Let us write

$$K_n = \sum_{i=1}^{n} I(f_i^0 \neq 0).$$

Then, given vectors $x_1, \ldots, x_n \in \mathscr{X}$, we have

$$\begin{aligned}\Big\|\sum_{i=1}^{n} f_i^0 x_i\Big\|_p^p &\geqslant \mathrm{E}\Big\|\sum_{i=1}^{n} f_i^0 x_i\Big\|^p I(K_n = 1)\\ &= \sum_{i=1}^{n} \mathrm{E}(f_i^0)^p I(f_i^0 \neq 0,\ f_j^0 = 0 \text{ for } j \neq i)\,\|x_i\|^p\\ &\geqslant \sum_{i=1}^{n} \|x_i\|^p \sum_{j=1}^{n} (1-a_j)/(1-a_i) \geqslant a \sum_{i=1}^{n} \|x_i\|^p,\end{aligned}$$

which completes the proof of Lemma 3.

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Let $\{f_n\}$ and $\{x_n\}$ be as in (ii) and let

$$\sup_n \Big\|\sum_{i=1}^{n} \mathrm{E} f_i x_i\Big\| < \infty.$$

Now the martingale

$$X_n = \sum_{i=1}^{n} (\mathrm{E} f_i - f_i) x_i, \qquad n \in N,$$

has the positive part $\{X_n^+\}$ bounded in $L_1(\mathscr{X})$, hence X_n itself is L_1-bounded by hypothesis. Therefore,

$$\sup_n \Big\|\sum_{i=1}^{n} f_i x_i\Big\|_1 < \infty.$$

(ii) $\Rightarrow$ (iii). It suffices to show that, for all $x_1, \ldots, x_n \in \mathscr{X}$,

$$\sum_{i=1}^{n} \|x_i\| \leqslant C \Big\|\sum_{i=1}^{n} |x_i|\Big\| \tag{4}$$

for some constant $C > 0$. Then we put

$$\|x\|_* = \sup\Big\{\sum_{i=1}^{n} \|x_i\| \colon x_1, \ldots, x_n \in \mathscr{X} \text{ and } \sum_{i=1}^{n} |x_i| \leqslant |x|\Big\}.$$

Then $\|\cdot\|_*$ is the complete monotone norm equivalent to the original one and additive for positive elements from $\mathscr{X}$ (cf. [5], p. 242, see also [6], Lemma 2). Hence the identity operator on $\mathscr{X}$ is the required isomorphism from $\mathscr{X}$ onto an AL-space. Now (4) holds by Lemmas 2 and 3 and by the Closed Graph Theorem. Thus (ii) $\Rightarrow$ (iii) is proved.

(iii) $\Rightarrow$ (i). Now, if $\|\cdot\|$ denotes an AL-norm, then by a standard argument (one can apply the fact that the set of simple functions is dense in $L_1(\mathscr{X})$) we have

$$\mathbf{E}\,\|X\| = \|\mathbf{E}\,|X|\| \quad \text{for } X \in L_1(\mathscr{X}).$$

Thus, if $\{X_n\}$ is a submartingale with $\sup_n \mathbf{E}\,\|X_n^+\| < \infty$, then

$$\sup_n \mathbf{E}\,\|X_n\| \leqslant 2 \sup_n \mathbf{E}\,\|X_n^+\| + \mathbf{E}\,\|X_1\| < \infty,$$

since $|x| = 2x^+ - x$ for $x \in \mathscr{X}$ and since $\mathbf{E}\,X_n \geqslant \mathbf{E}\,X_1$. This completes the proof of Theorem 1.

Remark 4. In the context of Theorem 4.1 in [5] there arises a question concerning the behaviour of p-integrable submartingales with

$$\sup_n \mathbf{E}\,\|X_n^+\|^p < \infty.$$

Proposition 3 can easily be formulated and proved after changing the power 1 into p ($p > 1$). However, even on the real line there exists a submartingale with $\sup_n \mathbf{E}\,\|X_n^+\|^p < \infty$ (or a positive p-integrable martingale, etc.), but, at the same time,

$$\sup_n \mathbf{E}\,\|X_n\|^p = \infty.$$

To see this it suffices to put

$$X_n = \sum_{i=1}^{n} \big(i^{-p} - iI(S_i)\big),$$

where $\{S_n\}$ is a sequence of independent events such that $P(S_n) = i^{-p-1}$.

5. Cone of absolutely summing mappings. Let $\mathscr{X}$ be a Banach lattice and let $\mathscr{Y}$ be a Banach space. A linear operator $T\colon \mathscr{X} \to \mathscr{Y}$ is said to be *cone absolutely summing* (c.a.s.) if, for any summable positive sequence $\{x_n\}$, $\{Tx_n\}$ is absolutely summable (cf. [5], Chapter IV.3, for the properties of c.a.s. operators used in the sequel).

THEOREM 2. *Let $T\colon \mathscr{X} \to \mathscr{Y}$ be a linear continuous operator. Then the following statements are equivalent:*

(i) *T is c.a.s.*

(ii) *For any $\mathscr{X}$-valued submartingale $\{X_n\}$ such that*

$$\sup_n \|\mathrm{E} X_n^+\| < \infty$$

$\{TX_n\}$ is bounded in $L_1(\mathscr{Y})$.

(iii) *For any $\mathscr{X}$-valued submartingale $\{X_n\}$ such that*

$$\sup_n \mathrm{E}\|X_n^+\| < \infty$$

$\{TX_n\}$ is bounded in $L_1(\mathscr{Y})$.

(iv) *For any increasing positive sequence $\{A_n\} \subset L_1(\mathscr{X})$ such that*

$$\sup_n \|\mathrm{E} A_n\| < \infty$$

$\{TA_n\}$ converges both a.s. and in $L_1(\mathscr{Y})$.

Moreover, if $\mathscr{X}$ is a KB*-space, then any of statements* (i)-(iv) *is equivalent to*

(v) *For any positive (regular) martingale $\{X_n\}$, taking values in $\mathscr{X}$, $\{TX_n\}$ is bounded in $L_1(\mathscr{Y})$.*

Proof. (i) $\Rightarrow$ (ii). Let $\{X_n\} \subset L_1(\mathscr{X})$ be a submartingale and let

$$\sup_n \|\mathrm{E} X_n^+\| < \infty.$$

Since T is c.a.s. iff there exists an $x^* \in \mathscr{X}^*$ ($\mathscr{X}^*$ denotes the norm dual of $\mathscr{X}$) such that for all $x \in \mathscr{X}$

$$\|Tx\| \leqslant \langle |x|, x^* \rangle,$$

we have

$$\mathrm{E}\|TX_n\| \leqslant \mathrm{E}\langle |X_n|, x^* \rangle = \langle \mathrm{E}|X_n|, x^* \rangle \leqslant 2\|x^*\| \sup_n \|\mathrm{E} X_n^+\| < \infty.$$

(ii) $\Rightarrow$ (iii) follows immediately.

(iii) $\Rightarrow$ (i) ((iv) $\Rightarrow$ (i)). Let $\{x_n\}$ be a positive summable sequence, $x_n \in \mathscr{X}$. Put

$$A_n = \sum_{i=1}^n f_i^0 x_i \quad \text{and} \quad X_n = \sum_{i=1}^n x_i - A_n,$$

where f_n^0, $n = 1, 2, \ldots$, are defined as in the proof of Lemma 3. Then $\{A_n\}$ is a positive increasing sequence with

$$\mathrm{E} A_n = \sum_{i=1}^n x_i,$$

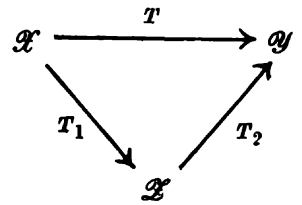
and $\{X_n\}$ is a martingale with $\sup_n \mathbf{E}\|X_n^+\| < \infty$. Now, by hypothesis,

$$\sup_n \mathbf{E}\Big\| \sum_{i=1}^{n} f_i^0 x_i \Big\| < \infty.$$

Thus $\{Tx_n\}$ is absolutely summable, since by Lemma 3

$$\sum_{i=1}^{n} \|Tx_i\| \leqslant \text{const} \cdot \sup_n \mathbf{E}\Big\| \sum_{i=1}^{n} f_i^0 x_i \Big\|.$$

(i) $\Rightarrow$ (iv). T is c.a.s. iff there exists an AL-space $\mathscr{Z}$ such that the diagram



commutes, where $T_1 \geqslant 0$ and T_2 are continuous linear operators. Now, if $\{A_n\}$ increases and $\sup_n \|\mathbf{E}A_n\| < \infty$, then

$$\mathbf{E}\|T_1 A_n\| = \|\mathbf{E}T_1 A_n\| \leqslant \|T_1\| \sup_n \|\mathbf{E}A_n\| < \infty.$$

Since each AL-space is a KB-space, by Lemma 1 the sequence $\{T_1 A_n\}$ converges both a.s. and in $L_1(\mathscr{Z})$, so also does $\{TA_n\} = \{T_2 T_1 A_n\}$.

To show (i) $\Rightarrow$ (v) $\Rightarrow$ (iii) we assume that $\mathscr{X}$ is a KB-space.

(i) $\Rightarrow$ (v) follows, since

$$\mathbf{E}\|TX_n\| \leqslant \langle \mathbf{E}X_1, |x^*| \rangle \quad \text{for some } x^* \in \mathscr{X}^*,$$

where $\{X_n\}$ is a positive martingale and T is c.a.s.

(v) $\Rightarrow$ (iii). Let $\{X_n\}$ be a submartingale with

$$\sup_n \mathbf{E}\|X_n^+\| < \infty.$$

$\{X_n\}$ can be decomposed in the form $\boldsymbol{X} = \boldsymbol{M} + \boldsymbol{A}$, where $\boldsymbol{M} = \{M_n\}$ is a martingale and $\boldsymbol{A} = \{A_n\}$ is an increasing positive sequence (cf. [4], p. 145, see also [7], p. 467). Since $\{A_n\}$ is positive, we have

$$\sup_n \mathbf{E}\|M_n^+\| \leqslant \sup_n \mathbf{E}\|X_n^+\| < \infty.$$

Thus, by Proposition 1, $\boldsymbol{M} = \boldsymbol{M}^+ - \boldsymbol{M}^-$ and, consequently, by hypothesis,

$$\sup_n \mathbf{E}\|TM_n\| < \infty.$$

To finish the proof it suffices to note that

$$A_n \leqslant X_n^+ + M_n^- = X_n^+ + M_n^+ - M_n^- \leqslant 2X_n^+ - M_n,$$

so

$$\mathrm{E}\,\|TA_n\| \leqslant 2\,\|T\|\,\mathrm{E}\,\|X_n^+\| + \mathrm{E}\,\|TM_n\|.$$

Thus the proof is complete.

COROLLARY. *Let $T\colon \mathscr{X} \to \mathscr{Y}$ be a c.a.s operator and assume that $\mathscr{Y}$ has the Radon-Nikodym property. If $\{X_n\}$ is a submartingale satisfying the condition*

$$\sup_n \|\mathrm{E}X_n^+\| < \infty,$$

then there exists a $Y \in L_1(\mathscr{Y})$ such that $TX_n \to Y$ a.s.

Proof. Let us decompose $\{X_n\}$ by the Doob formula: $X_n = M_n + A_n$. By hypothesis we have

$$\sup_n \|\mathrm{E}M_n^+\| < \infty \quad \text{and} \quad \sup \|\mathrm{E}A_n\| < \infty.$$

Thus, by Theorem 2 ((ii) and (iv)) and by the Chaterji Theorem [1], $\{TM_n\}$ converges a.s. and $\{TA_n\}$ converges a.s., since $\{TM_n\}$ is the martingale bounded in $L_1(\mathscr{Y})$.

The Corollary generalizes Theorem 4.2 in [7].

Remark 5. The terms of type "for any martingale..." etc. are meant in the sense "for any probability space and for any martingale defined on it..." etc. However, one can assume a probability space to be sufficiently "large" only because of the used constructions.

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Reçu par la Rédaction le 10. 5. 1977;
en version modifiée le 4. 10. 1977