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MORITA EQUIVALENCE OF ALGEBRAIC THEORIES

BY

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Algebraic theories R and T are said to be *Morita equivalent* if the corresponding algebraic categories $\mathrm{Alg}(R)$ and $\mathrm{Alg}(T)$ are equivalent as categories. Morita equivalence in this sense has been characterized in various ways for several special cases; some of the results of Banaschewski [1], Hu [5], Knauer [6], Morita [9] (see Cohn [2] for an exposition), and Wraith [11] can be construed as providing such characterizations. The main result of this paper*, Theorem 2.6, is a syntactical characterization of all the algebraic theories R which are Morita equivalent to a given algebraic theory T.

1. Notation and definitions. An *algebraic theory* is a locally small category T together with a product-preserving functor J_T: $\mathrm{Card}^{\mathrm{op}} \to T$ which determines a bijective correspondence between the objects of T and the cardinal numbers, which are identified with the objects of a skeleton Card of the category Set of sets. It follows that every object of T is a power of one particular "base" object $X = J_T(1)$ relative to a distinguished cone of projections indexed by J_T. A *T-algebra* in a category M is a product-preserving functor A: $T \to M$; the full subcategory of Set^T whose objects are the T-algebras in Set is $\mathrm{Alg}(T)$. The underlying-set functor U_T: $\mathrm{Alg}(T) \to \mathrm{Set}$ has the free T-algebra functor F_T: $\mathrm{Set} \to \mathrm{Alg}(T)$ as its left adjoint. The basics of this style of infinitary universal algebra are presented in [11].

An object A in a category M is said to be *tractable* if there is an algebraic theory T and a full and faithful T-algebra 'A': $T \to M$ for which A = 'A'(X); in that case, T is called the *algebraic structure* of A, and 'A' is called the *structure algebra* of A. Note that, for every algebraic theory T, the underlying-set functor U_T is a tractable object in $\mathrm{Set}^{\mathrm{Alg}(T)}$ whose algebraic structure is T.

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* The results reported in this paper appear in the author's Ph. D. Thesis [3], written under the supervision of Prof. A. H. Lachlan.

2. Morita equivalence of algebraic theories. Our starting point for discussing Morita equivalence is the consideration of equivalence functors $\mathrm{Alg}(T) \to \mathrm{Alg}(R)$. It will be helpful to recall some basic facts about algebra-valued functors; these results follow from elementary category-theoretic considerations and are closely related to the structure-semantics adjointness discussed in [8], [11], and – for finitary universal algebra – in [7].

Every algebra-valued functor of the form $G: M \to \mathrm{Alg}(R)$ corresponds to an R-algebra $R \to \mathrm{Set}^M$ whose "underlying object" is the set-valued component $U_R \cdot G$ of G. Conversely, every R-algebra $R \to \mathrm{Set}^M$ whose underlying object is U determines a functor $M \to \mathrm{Alg}(R)$ whose set-valued component is U. In particular, if U is a tractable object in Set^M whose algebraic structure is R, then the functor $E_U: M \to \mathrm{Alg}(R)$ determined by the structure algebra 'U': $R \to \mathrm{Set}^M$ is called the *comparison functor* for U. For example, the identity functor $\mathrm{Alg}(R) \to \mathrm{Alg}(R)$ is the comparison functor for U_R.

It is not difficult to see that an equivalence functor $E: M \to \mathrm{Alg}(R)$ is the comparison functor for its own set-valued component $U_R \cdot E$, whose algebraic structure is R. Thus, the problem of characterizing all the algebraic theories R which are Morita equivalent to a given algebraic theory T reduces to the following two problems:

2.1. Characterize those tractable functors U: $\mathrm{Alg}(T) \to \mathrm{Set}$ whose comparison functor E_U is an equivalence functor.

2.2. For every such functor U, describe the algebraic structure R of U.

Our first result provides a solution to Problem 2.1. Objects A and B in a category M are said to be *retract-equivalent* if, for some cardinals m and n, the powers A^n and B^m exist, and A is a retract of B^m while B is a retract of A^n.

2.3. THEOREM. *Let T be an algebraic theory and let U*: $\mathrm{Alg}(T) \to \mathrm{Set}$ *be a tractable functor. Then the comparison functor E_U is an equivalence functor if and only if U and U_T are retract-equivalent in* $\mathrm{Set}^{\mathrm{Alg}(T)}$.

Proof. Is is easy to see that if E_U is an equivalence functor, then U is represented by a T-algebra A which has the same category-theoretic properties as a free algebra; this is because $E_U(A)$ is a free algebra. In particular, A is dually retract-equivalent to the free T-algebra $F_T(1)$, which means that U is retract-equivalent to U_T.

Assuming now that U and U_T are retract-equivalent, we shall use the main result of [8] to prove that E_U is an equivalence functor. According to Linton's theorem, E_U is an equivalence functor if and only if U has a left adjoint and preserves and reflects congruence relations and regular epimorphisms. In both $\mathrm{Alg}(T)$ and Set it happens that the congruence relations are the equivalence relations, i.e., the binary relations which are reflexive, symmetric, and transitive; and the regular epimorphisms are the surjective homomorphisms and the surjective functions, respectively. Suppose that $r\colon U_T^n \to U$ and $g\colon U^m \to U_T$ are retractions; note that every component of r or g is a retraction in Set, so must be a surjection.

The functor U is representable, since it is a retract of the representable functor U_T^n. Representable functors $\mathrm{Alg}(T) \to \mathrm{Set}$ have left adjoints, so U has a left adjoint.

To show that U preserves regular epimorphisms, suppose that $h\colon A \to B$ is a surjective homomorphism; then $U_T(h)$ is a surjective function, so $U_T^n(h)$ is surjective. Then $U(h) \cdot r_A = r_B \cdot U_T^n(h)$, where the right-hand side is surjective, so $U(h)$ is surjective. Now we shall show that U reflects regular epimorphisms. If $h\colon A \to B$ is such that $U(h)$ is a surjective function, then $U^m(h)$ is also surjective. But then $U_T(h) \cdot g_A = g_B \cdot U^m(h)$, where the right-hand side is surjective, so $U_T(h)$ is surjective, i.e., h is a surjective homomorphism.

Since U is representable, it preserves kernel pairs; hence it preserves congruence relations. The proof that U reflects congruence relations is as follows. As remarked above, it is sufficient to show that U reflects equivalence relations. Given a binary relation E in $\mathrm{Alg}(T)$ such that $U(E)$ is an equivalence relation in Set, it is easy to verify that $U^m(E)$ must also be an equivalence relation. But then $U_T(E)$, as a retract of the equivalence relation $U^m(E)$, is also an equivalence relation. Since U_T reflects equivalence relations, E is an equivalence relation in $\mathrm{Alg}(T)$.

Theorem 2.3 is a variant of a result which is apparently part of the unpublished "folklore" of category theory: E_U is an equivalence functor if and only if $U\colon \mathrm{Alg}(T) \to \mathrm{Set}$ is represented by a regular progenerator, i.e., by a T-algebra which is dually retract-equivalent to the free T-algebra $F_T(1)$. See, e.g., Lawvere's remarks in [7], p. 86 ff., and a comment by Wraith in [11], p. 54. This "folklore" result is a natural generalization of the Morita theorem of module theory (see [2] and [9]). Hu's theorem characterizing the category of Boolean algebras as a finitary algebraic category (see [5]) is derivable from the "folklore" version of Theorem 2.3, as is Wraith's result in [11] that the matrix theories of an algebraic theory T are Morita equivalent to T.

With the next two lemmas we shall solve Problem 2.2. Recall that if B is a retract of A, with $r\colon A \to B$ and $s\colon B \to A$ such that $r \cdot s = \mathrm{id}_B$, then $u = s \cdot r$ is idempotent, i.e., $u \cdot u = u$. On the other hand, if $u\colon A \to A$ is idempotent and factors as $u = s \cdot r$, where $r\colon A \to B$ is an epimorphism and $s\colon B \to A$ is a monomorphism, then in fact r is a retraction and s is a coretraction. The retract B is called an *image* of u.

2.4. LEMMA. *Let A and B be objects in a category M which has products. Then A and B are retract-equivalent, with A being a retract of B^m and B being a retract of A^n, if and only if there are arrows $u\colon A^n \to A^n$, $d\colon A \to A^{n\times m}$, and $p\colon A^{n\times m} \to A$ such that $u \cdot u = u$, $u^m \cdot d = d$, $p \cdot d = \mathrm{id}_A$, and B is an image of u.*

Proof. If $u \cdot u = u$ and B is an image of u, then – as was pointed out above – there are r: $A^n \to B$ and s: $B \to A^n$ with $s \cdot r = u$ and $r \cdot s = \mathrm{id}_B$. If d and p are given as described, then it is easy to verify that $p \cdot s^m$: $B^m \to A$ is a retraction with coretraction $r^m \cdot d$: $A \to B^m$.

Now suppose that A is a retract of B^m and B is a retract of A^n. Then we have r: $A^n \to B$ and s: $B \to A^n$ with $r \cdot s = \mathrm{id}_B$, and we also have g: $B^m \to A$ and h: $A \to B^m$ with $g \cdot h = \mathrm{id}_A$. The arrows called for by the lemma are $u = s \cdot r$, $d = s^m \cdot h$, and $p = g \cdot r^m$.

Given an algebraic theory T with an idempotent operation u: $X^n \to X^n$, we define a new algebraic theory $T|u$, the restriction of T to u, as follows. Say that an operation g: $X^{n \times k} \to X^{n \times j}$ of T is a *u-operation* if $u^j \cdot g \cdot u^k = g$. Note that a composite of u-operations is a u-operation. Then $T|u$ is the category whose arrows are the u-operations of T and whose identity arrows are the powers (computed in T) of the operation u. The product-indexing functor J: $\mathrm{Card}^{\mathrm{op}} \to T|u$ sends each arrow f: $j \to k$ of Card to $u^j \cdot J_T(f) \cdot u^k$. The verification that $T|u$ is actually an algebraic theory is contained in the next lemma.

2.5. LEMMA. *Let A be a tractable object whose algebraic structure is T and let B be the image of an idempotent arrow u: $A^n \to A^n$. Then B is a tractable object whose algebraic structure is $T|u$.*

Proof. There are r: $A^n \to B$ and s: $B \to A^n$ with $s \cdot r = u$ and $r \cdot s = \mathrm{id}_B$. It is obvious that a retract of a tractable object is tractable. Let R be the algebraic structure of B. For each R-operation g: $B^k \to B^j$, let $t(g) = s^j \cdot g \cdot r^k$; it is easy to verify that this defines an isomorphism of categories t: $R \to T|u$ such that $t \cdot J_R = J$.

The main theorem follows from 2.3–2.5.

2.6. THEOREM. *Algebraic theories R and T are Morita equivalent if and only if, for some cardinals m and n, there are T-operations u: $X^n \to X^n$, d: $X \to X^{n \times m}$, and p: $X^{n \times m} \to X$ such that $u \cdot u = u$, $u^m \cdot d = d$, $p \cdot d = \mathrm{id}_X$, and $R \cong T|u$.*

If R and T are finitary and Morita equivalent, then the cardinals m and n mentioned in Theorem 2.6 may be taken to be finite.

2.7. COROLLARY. *Retract-equivalent tractable objects in any category have Morita equivalent equational structures.*

3. Examples and related results. In this section we show that Theorem 2.6 is a direct generalization of the characterization of Morita equivalent monoids given by Banaschewski [1] and Knauer [6]. Two well-known representations of m-valued Post algebras are shown to provide the data required by Theorem 2.6 to prove that the algebraic theory of m-valued Post algebras is Morita equivalent to the algebraic theory of Boolean algebras. Finally, we discuss the relationship between Theorem 2.6 and a result of Elkins and Zilber [4] characterizing Morita equivalent small categories.

Small categories M and N are said to be *Morita equivalent* if the functor categories Set^M and Set^N are equivalent. Banaschewski and Knauer have found nearly identical necessary and sufficient conditions for monoids M and N (represented as one-object categories) to be Morita equivalent as small categories.

3.1. THEOREM (Banaschewski [1] and Knauer [6]). *Monoids M and N are Morita equivalent if and only if there are elements u, v, w of N such that $uu = u$, $uw = w$, $vw = e$, and $M \cong uNu$.*

Every monoid N is naturally associated with an algebraic theory $\bar{N}$ whose nontrivial operations are all unary and form a monoid (under composition) isomorphic to N. It is easy to see that the functor category Set^N is equivalent to the algebraic category $\mathrm{Alg}(\bar{N})$. The conditions given in Theorem 3.1 are evidently the same as those given in Theorem 2.6, except that in Theorem 3.1 we have $m = n = 1$. Thus, to obtain Theorem 3.1 from Theorem 2.6 it suffices to demonstrate that if there are any operations u, d, and p of $\bar{N}$ satisfying the conditions of Theorem 2.6, with $\bar{M} \cong \bar{N}|u$, then there are unary operations u', d', and p' of $\bar{N}$, which satisfy the conditions of Theorem 2.6, such that $\bar{M} \cong \bar{N}|u'$. This, however, follows easily from the fact that all of the nontrivial operations of $\bar{N}$ are unary.

For all finite $m > 1$, the algebraic theory P_m of m-valued Post algebras is known to be Morita equivalent to the algebraic theory BA of Boolean algebras. A survey of results concerning Post algebras is given in [10], Chapter 7, where a presentation of P_m is described in terms of:

(i) constants $e_0, e_1, \ldots, e_{m-1}$;

(ii) unary operations $\neg, D_1, D_2, \ldots, D_{m-1}$;

(iii) binary operations $\wedge, \vee, \Rightarrow$;

(iv) a list of equational axioms (p_0), (p_1), ..., (p_8).

The equational axioms ensure that every m-valued Post algebra is a Heyting algebra with respect to e_0, e_{m-1}, $\neg$, $\wedge$, $\vee$, and $\Rightarrow$, with e_0 being the smallest element and e_{m-1} being the largest one. The operation D_1 coincides with "double negation", i.e., $D_1(x) = \neg\neg x$ (see [10], p. 137). The P_m-operations u, d, and p required by Theorem 2.6 to witness the Morita equivalence of P_m and BA may be defined as follows:

(i) u is the operation $X \to X$ given by

$$u(x) = \neg\neg x;$$

(ii) d is the operation $X \to X^{m-1}$ given by

$$d(x) = (D_1(x), D_2(x), \ldots, D_{m-1}(x));$$

(iii) p is the operation $X^{m-1} \to X$ given by

$$p(x_1, x_2, \ldots, x_{m-1}) = \bigvee_{0<i<m} (e_i \wedge x_i).$$

Axiom (p_5), which states that $D_i(D_j(x)) = D_j(x)$ for all i and j, guarantees that $u = D_1$ is idempotent and that $u^{m-1} \cdot d = d$. Axiom (p_7) states precisely that $p \cdot d = \mathrm{id}_X$. The operation u picks out the set of all complemented elements in any m-valued Post algebra, and the u-operations of P_m are precisely the operations which preserve complemented elements. Proposition 1.4 on p. 136 of [10] states that the complemented elements of a Post algebra form a Boolean algebra with respect to the operations which preserve them: this corresponds to the condition $BA \cong P_m|u$.

The representation of the m-valued Post algebras as lattices of nonincreasing $(m-1)$-element chains in Boolean algebras (see [10], pp. 143 and 144, for details) also illustrates how Theorem 2.6 works. In this case, u, d, and p are Boolean operations:

(i) u is the idempotent operation $X^{m-1} \to X^{m-1}$ given by

$$u(x_1, x_2, \ldots, x_{m-1}) = (x_1, x_1 \wedge x_2, \ldots, x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1});$$

(ii) d is the diagonal operation $X \to X^{m-1}$ given by

$$d(x) = (x, x, \ldots, x);$$

(iii) p is the projection operation $X^{m-1} \to X$ given by

$$p(x_1, x_2, \ldots, x_{m-1}) = x_1.$$

According to Theorem 3.6 of [4], for any small categories M and N, the functor categories Set^M and Set^N are equivalent if and only if M and N have equivalent idempotent completions. An *idempotent completion* of a category M, according to Corollary 3.4 of [4], is a category $I(M)$, containing M as a full subcategory, such that every object of $I(M)$ is an image of an idempotent arrow of M and such that every idempotent arrow of M has an image in $I(M)$.

3.2. THEOREM. *Algebraic theories are Morita equivalent if and only if they have equivalent idempotent completions.*

Proof. If R and T are Morita equivalent algebraic theories, then we have T-operations u, d, and p as described by Theorem 2.6, so $R \cong T|u$ is equivalent to a full subcategory of $I(T)$. Our Lemma 2.4 and Theorem 3.8 of [4], which does not require smallness of the categories concerned, are sufficient to show that $I(T)$ is equivalent to $I(R)$.

If R and T are algebraic theories with $I(R)$ equivalent to $I(T)$, then by Theorem 3.8 of [4] it follows that R and T can be embedded in a category C such that the base objects of R and T are retract-equivalent in C. Here it should be noted that the embeddings can be assumed to be full and to preserve products, since C can be chosen to be equivalent to $I(T)$. By our Corollary 2.7, R and T are Morita equivalent.

It is clear from Theorem 2.6 that Morita equivalent algebraic theories do not differ in many logically interesting ways. An isomorphism $R \cong T|u$ can be seen as a kind of interpretation of R in T; such interpretations of theories are shown in [3] to give rise to an important class of algebra-valued functors which includes Boolean power constructions, viewed as algebra-valued functors defined on various algebraic categories of Boolean algebras. Because of the amount of logical machinery involved, the applications of Theorem 2.6 to the logical analysis of algebra-valued functors are beyond the scope of this paper and will be published separately.

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