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ON SETS OF WEAK UNIFORM DISTRIBUTION

BY

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In this paper* we characterize those sets A of natural numbers for which there is an arithmetical function which is weakly uniformly distributed modulo m if and only if $m \in A$, thus solving a problem posed by W. Narkiewicz.

Let f be an integer-valued arithmetical function and write

$$N(f, a, m, x) = \# \{n\colon n \leq x, f(n) \equiv a \pmod{m}\}.$$

We say that f is *weakly uniformly distributed* (w.u.d.) modulo m if for arbitrary a, b *prime* to m we have

$$N(f, a, m, x) \sim N(f, b, m, x) \quad \text{as } x \to \infty.$$

This concept was introduced and investigated (especially for multiplicative functions) by W. Narkiewicz in a series of papers (cf., e. g., [1] and [2]). The following problem has also been posed by him.

We define the *set of weak uniform distribution* of a function f as the set of all natural numbers m for which f is w.u.d. modulo m. The problem is to describe those sets A that can occur as a set of weak uniform distribution. The analogous problem for the ordinary uniform distribution was solved by Zame [3]. He found the condition that together with any number A must contain all its divisors.

A similar answer can be given in the case of weak uniform distribution. We say that d is a *close divisor* of m (m is a *close multiple* of d) if $d \mid m$ and d is divisible by every prime factor of m.

THEOREM. *Let A be a set of natural numbers. There exists an arithmetical function whose set of weak uniform distribution is A if and only if $1 \in A$, $2 \in A$ and for all $n \in A$ all the close divisors of n also belong to A.*

The necessity of the conditions is obvious; in what follows we construct the function f for a set A satisfying the conditions of the Theorem.

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The functions will be such that each set

$$\{n\colon f(n) \equiv a \pmod{m}\}$$

will have an asymptotic density $\delta(a, m) > 0$.

LEMMA 1. *Given a function $\delta(a, m)$, for the existence of an arithmetical function f such that*

$$N(f, a, m, x) = \delta(a, m)x + o(x)$$

for all a and m the following conditions are necessary and sufficient:

(i) $\delta(a, m) \geqslant 0$, $\delta(0, 1) = 1$;

(ii) $\delta(a, m)$ *is periodic in a with a period m;*

(iii) *for all a, m and d,*

$$\sum_{j=0}^{d-1} \delta(a+jm, dm) = \delta(a, m). \tag{1}$$

If these conditions are satisfied, we call δ a *distribution function.*

Proof. The necessity of the conditions is clear.

If δ satisfies (i)–(iii), then f can be constructed, e.g., as follows. Let $w(k)$ be an integer-valued function tending to infinity so slowly that $w(k)! = o(k)$. Now on the $2k+1$ values $k^2 \leqslant n < (k+1)^2$ let f assume the value j, $1 \leqslant j \leqslant w(k)!$,

$$[2\delta(j, w(k)!)k]$$

times (and define it arbitrarily on the remaining at most $2w(k)!+1 = o(k)$ numbers). The easy proof that its distribution is really δ is left to the reader.

Now to prove the Theorem it is sufficient to construct $\delta(a, m)$ so that it be constant on the numbers a coprime to m if $m \in A$ (in this case we call δ also w.u.d. modulo m) and not constant otherwise.

First we construct, for any number $k \geqslant 2$, an auxiliary "perturbing function" $\mu_k(a, d)$ that satisfies (ii) and (iii) of Lemma 1 but $\mu_k(0, 1) = 0$, and which is not w.u.d. modulo k (and hence neither modulo the close multiples of k) but is w.u.d. modulo all the other integers. The final δ will be given as a sum of these μ_k's for $k \notin A$ (with weights) and a weak uniform distribution to provide positivity.

Let k, m be natural numbers, $m = m_1 m_2$, where m_1 is composed of the prime factors of k and $(m_2, k) = 1$. For any integer a we put

$$\mu_k(a, m) = \begin{cases} \dfrac{1}{m_1} \sin \dfrac{2\pi a}{k} & \text{if } m_2 \mid a \text{ and } k \mid m_1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

LEMMA 2. *The functions μ_k satisfy* (ii) *and* (iii) *of Lemma* 1; *moreover, $\mu_k(a, m) = 0$ whenever $(a, m) = 1$ and m is not a close multiple of k.*

Proof. Periodicity is obvious. If $(a, m) = 1$ and $\mu_k(a, m) \neq 0$, then $m_2 | a$ yields $m_2 = 1$, and then $k | m = m_1$ just means that m is a close multiple of k.

We have to prove (iii). Let $d = d_1 d_2$, where into d_1 we put the primes that occur in k and $(k, d_2) = 1$. If $m_2 \nmid a$, then also $m_2 \nmid a+jm$, thus all the summands in (1) are 0. Similarly, if $k \nmid d_1 m_1$, then also $k \nmid m_1$ and again all the summands vanish as well as the right side. Hence we may assume that $m_2 | a$ and $k | d_1 m_1$.

Now $\mu_k(a+jm, dm) = 0$ except $m_2 d_2 | a+jm$. This means that

$$d_2 \left| \frac{a}{m_2} + jm_1, \quad jm_1 \equiv -\frac{a}{m_2} \pmod{d_2}. \right.$$

Since $(m_1, d_2) = 1$, this congruence has a unique solution $j^* \pmod{d_2}$, and then all the admissible values of j are $j^* + ld_2$, $0 \leqslant l \leqslant d_1 - 1$. Hence

$$\begin{aligned} \sum_{j=0}^{d-1} \mu_k(a+jm, dm) &= \sum_{l=0}^{d_1-1} \mu_k(a+j^* m + ld_2 m, dm) \\ &= \frac{1}{m_1 d_1} \sum_{l=0}^{d_1-1} \sin 2\pi \frac{a+j^* m + ld_2 m}{k}. \end{aligned}$$

This is 0 if $k \nmid m$ since $(k, d_2) = 1$, and it is

$$\frac{1}{m_1 d_1} d_1 \sin 2\pi \frac{a}{k}$$

if $k | m$, in both cases it is equal to $\mu_k(a, m)$ as wanted.

Proof of the Theorem. Consider also the distributions

$$\nu_k(a, m) = \begin{cases} 1/m_1 & \text{if } m_2 | a, \\ 0 & \text{otherwise}, \end{cases}$$

where m_1 and m_2 mean the same as in (2). It is easy to see that ν_k is a weak uniform distribution and always

$$|\mu_k(a, m)| \leqslant \nu_k(a, m). \tag{3}$$

Let $c_0, c_1, \ldots$ be any sequence of positive numbers whose sum is 1 and put

$$\delta(a, m) = \frac{c_0}{m} + \sum_{k=1}^{\infty} c_k \nu_k(a, m) + \sum_{k \in A} c_k \mu_k(a, m). \tag{4}$$

δ will clearly be a distribution function (positivity follows from (3)).

Now, if $m \in A$ and $k \notin A$, then k cannot be a close divisor of m by the assumption on A; hence by Lemma 2 we have $\mu_k(a, m) = 0$ whenever $(a, m) = 1$. Therefore the third term in (4) vanishes, which shows that δ is w.u.d. modulo m.

On the other hand,

$$\mu_k(1, m) - \mu_k(-1, m) = 0 \quad \text{or} \quad \frac{2}{m_1}\sin\frac{2\pi}{k} \geqslant 0,$$

which is always nonnegative, and is strictly positive if $k \geqslant 3$. Thus

$$\delta(1, m) > \delta(-1, m)$$

whenever $k = m$ occurs in the third term of (4), i.e., for $m \notin A$, and this shows that δ is not w.u.d. modulo m if $m \notin A$.

Remark. Professor Narkiewicz called my attention to the fact that the concept of weak uniform distribution I defined at the beginning of the paper differs slightly from his. He calls a function f w.u.d. modulo m only if the asymptotic equality

$$N(f, a, m, x) \sim N(f, b, m, x)$$

is a proper one in the sense that

$$N(f, a, m, x) \to \infty \quad \text{as } x \to \infty. \tag{5}$$

In this case a function may not be w.u.d. modulo 2. For this case we have the following

THEOREM*. *If the condition* (5) *is incorporated in the definition of the weak uniform distribution, then the Theorem holds in the following modified form: instead of* $2 \in A$ *we require that either* $2 \in A$ *or no even number belongs to* A.

Proof. $2 \notin A$ is possible only if (5) does not hold for $m = 2$ and $a = 1$, i.e., all but a finite number of the values of f are even. Then (5) also fails to hold for any even m and $a = 1$, and this shows the necessity of the condition.

Now we show the sufficiency. If $2 \in A$, then the same construction works as for the first concept of weak uniform distribution.

Assume this is not the case, i.e., all elements of A are odd. Let $A' = A \cup \{2\}$ and let f' be a function whose set of weak uniform distribution in the first sense is A' and $\delta(a, m) > 0$ for all m and a, as in our construction. Put $f(n) = 2f'(h)$. Clearly, f is always even, and hence not w.u.d. modulo any even number. If m is odd, then a multiplication by 2 obviously does not change the $(\bmod\, m)$ weak uniform distribution property. Since $\delta(a, m) > 0$, (5) is satisfied, and therefore the difference of the concepts plays no role.

I was informed that the same problem was also solved in a different way by Ms. Rosochowicz [1].

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[1] E. Rosochowicz, *On weak uniform distribution of sequences of integers*, this fasc., pp. 173–182. [Note of the Editors]

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