COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

MATHEMATICAL THEORY OF VON NEUMANN ECONOMIC MODELS

REPORT ON RECENT RESULTS

BY

JERZY ŁOŚ (WARSZAWA)

Von Neumann constructed his model over forty years ago, however, it was not until after his paper (1937) was published in an English translation (1945-1946) that economists took an interest in it [33], [34]. The second most important contribution to the theory seems still to be the paper by Kemeny et al. [14] which extended the original definition of an equilibrium and did make it more accessible both to economic and mathematical investigations. From those times on a great number of papers have been published considering either the von Neumann model itself or dealing with specific economies described by means of von Neumann models. Indeed, every description of an economy by means of linear processes with the same time difference between input and output uses a von Neumann model or, at least, can be converted into such a model.

In the last two decades the von Neumann model was subjected to many generalizations. By allowing matrices to have negative entries we obtain possibilities to express instant exchange processes (Mardoń [27]) as well to describe small economies acquiring rough materials on a market (Łoś [21]). By ordering vector spaces not by coordinates but by arbitrary cones we can introduce dependent processes, even infinitely many of them, we can also exclude some prices as not allowable and then extend the admissible (non-negative) bundles of commodities to involve those which, although have some negative coordinates, yet exhibit always positive values at allowable prices (Moeschlin [29], Kapitel 8, Ballarini and Moeschlin [1], Łoś [21]). By extending the number of matrices to three or four we express problems of consumption, labour and savings (Morgenstern and Thompson [30], Łoś [18], Ballarini and Moeschlin [1]), reswitching phenomena (Łoś and Łoś [23]), efficiency frontiers (Bromek [3], Łoś and Łoś [22]), and the list of possibilities is plausibly far from being exhausted.

This paper is a report on results concerning von Neumann models obtained recently in Warsaw, precisely in the years 1971-1976. Most of them have been published in the three volumes Łoś and Łoś [24], Łoś and Łoś [25] and Łoś et al. [26], some are still unpublished. We report also on results of other economists and mathematicians, but only if their work is closely related to the problems under scrutiny in Warsaw. We concentrate our attention on mathematical results giving few or none of the economic interpretations. No proofs are given here. They are either available in the literature or will be published soon.

1. Basic notions and notation. By R^n we denote the space of n-dimensional row vectors $x = (x_1, x_2, \ldots, x_n)$, while by $\tilde{R}^n$ we denote the space of n-dimensional column vectors

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \langle q_1, q_2, \ldots, q_n \rangle.$$

These spaces form a dual pair and

$$xq = \sum_{i=1}^{n} x_i q_i.$$

The same notation applies to other spaces, for instance, R^m with vectors $y = (y_1, y_2, \ldots, y_m)$ and $\tilde{R}^m$ with vectors

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} = \langle p_1, p_2, \ldots, p_m \rangle.$$

Any $(n \times m)$-matrix M with the left-multiplication serves as a transformation

$$\cdot M: R^n \to R^m$$

and with the right-multiplication as a transformation

$$M\cdot: \tilde{R}^m \to \tilde{R}^n.$$

We have $(xM)p = x(Mp)$. The transformations $\cdot M$ and $M\cdot$ are called *mutually dual.*

A *cone* (*convex cone*) in R^n is a subset S of R^n such that

$$S+S = \{x_1+x_2 \mid x_1, x_2 \in S\} \subset S$$

and

$$\lambda S = \{\lambda x \mid x \in S\} \subset S \quad \text{for all } \lambda \geqslant 0.$$

The set $S-S = \{x_1-x_2 \mid x_1, x_2 \in S\}$ is the subspace of R^n spanned over S. If $S-S = R^n$, the cone S is called *solid*. If $-x, x \in S$ implies $x = 0$, the cone S is called *pointed*. The cone S is polyhedral iff there exist a finite number of vectors $x_1, x_2, \ldots, x_k$ such that

$$S = \Big\{\sum_{i=1}^{k} \xi_i x_i \mid \xi_i \geqslant 0, \ i = 1, 2, \ldots, k\Big\}.$$

A polyhedral cone is closed in the usual topology of R^n. The dual cone of S is

$$S^* = \{q \in \tilde{R}^n \mid xq \geqslant 0 \text{ for all } x \in S\}.$$

The dual cone is always closed (as an intersection of closed half-spaces). The dual of the dual

$$S^{**} = \{x \in R^n \mid xq \geqslant 0 \text{ for all } q \in S^*\}$$

equals S iff S is closed. The dual of a polyhedral cone is polyhedral. The *non-negative orthant*

$${}^+R^n = \{x \in R^n \mid x_i \geqslant 0, \ i = 1, 2, \ldots, n\}$$

of R^n is a polyhedral cone and its dual is the non-negative orthant ${}^+\tilde{R}^n$ of $\tilde{R}^n$. The dual of a pointed cone is solid and vice versa.

For any cone S in R^n, defining

$$x^1 \leqslant x^2 \quad \text{if and only if} \quad x^2 - x^1 \in S,$$

we get a preordering relation in R^n (reflexive and transitive) which, moreover, preserves the algebraic operations in R^n, i.e. $x^1 \leqslant x^2$ implies $x^1 + x^3 \leqslant x^2 + x^3$ and $\lambda x^1 \leqslant \lambda x^2$ for all $\lambda > 0$. If S is pointed, then $\leqslant$ is an (partial) order, i.e. $x^1 \leqslant x^2$ and $x^2 \leqslant x^1$ imply $x^1 = x^2$. If S is solid, then there are elements $\bar{x}$ in S such that for every $x \in R^n$ there exists a number $\xi > 0$ with $x \leqslant \xi \bar{x}$. All such elements form the *interior* (in the sense of the metric topology in R^n) of S. We write $x^1 < x^2$ to denote that $x^2 - x^1$ is in the interior of S. This sign applies only if S is solid. If S is pointed, we write $x^1 \lneqq x^2$ to denote that $x^1 \leqslant x^2$ and $x^1 \neq x^2$. Thus $S = \{x \mid x \geqslant 0\}$ for all cones S, $\operatorname{int} S = \{x \mid x > 0\}$ for solid S (positive vectors) and $S \setminus \{0\} = \{x \mid x \gneqq 0\}$ for pointed S (semi-positive vectors).

If S is closed, then $x^{(k)} \leqslant x'^{(k)}$ for $k = 1, 2, \ldots$, and $x^{(k)} \to x$, $x'^{(k)} \to x'$ imply $x \leqslant x'$. If S is closed and pointed, then for every $\bar{x} \geqslant 0$ the set $\{x \geqslant 0 \mid x \leqslant \bar{x}\}$ is compact. Under the same assumptions, S^* is solid, thus there exists a $q > 0$ in S^* and the set $\{x \geqslant 0 \mid xq = 1\}$ is compact.

We call a pair (R^n, S) an *ordered space* (keeping in mind that it is only preordered if S is not pointed). A linear transformation between two ordered spaces $\cdot M\colon (R^n, S) \to (R^m, U)$ is *monotone* iff $x \geqslant 0$ implies $xM \geqslant 0$ (the first inequality means $x \in S$ and the second one means $xM \in U$).

We will always investigate such transformations with U closed, precisely with $U = T^*$, where T is a cone in $\tilde{R}^m$. By the *dual transformation* to $\cdot M$: $(R^n, S) \to (R^m, T^*)$ we shall understand then

$$M\cdot : (\tilde{R}^m, T) \to (\tilde{R}^n, S^*).$$

It is easily seen that the dual to a monotone transformation is also monotone.

If $\cdot M: R^n \to R^m$ and C is a cone in R^n, then $T = CM$, the image of C by $\cdot M$, is also a cone. If C is polyhedral, then CM is also polyhedral, but if C is closed but not polyhedral, then CM cannot be closed. It must be, however, an F_σ, that means a union of a sequence of closed sets. A theorem by Bromek and Kaniewski [7] states that:

If T is an F_σ-cone in R^m, then there exist a linear transformation $\cdot M$: $R^{m+1} \to R^m$ and a closed, pointed cone C in R^{m+1} such that $CM = T$.

2. Von Neumann models: definition. A *von Neumann model* consists of eight elements: a finite-dimensional linear space R^n, a pointed cone S in R^n, another finite-dimensional space $\tilde{R}^m$, a pointed cone T in $\tilde{R}^m$ and four matrices A_1, B_1, A_2, B_2 of dimension $n \times m$, which are understood as transformations either

$$\cdot A_1, \cdot B_1, \cdot A_2, \cdot B_2\colon (R^n, S) \to (R^m, T^*)$$

or

$$B_2\cdot, A_2\cdot, B_1\cdot, A_1\cdot : (\tilde{R}^m, T) \to (\tilde{R}^n, S^*).$$

Here $\tilde{R}^n$ and R^m are dual spaces to R^n and $\tilde{R}^m$, and T^* and S^* are dual cones (thus closed and solid) to T and S, respectively.

A von Neumann model can therefore be written as

$$\mathfrak{M} = (R^n, S, R^m, T, A_1, B_1, A_2, B_2).$$

The von Neumann model

$$\tilde{\mathfrak{M}} = (R^m, T, R^n, S, B_2, A_2, B_1, A_1)$$

is called the *dual* to $\mathfrak{M}$.

$R^n, R^m, \tilde{R}^n$ and $\tilde{R}^m$ are called spaces of *intensities*, of *commodities*, of *values* and of *prices*, respectively. The vectors in these spaces are denoted by x, y, q and p, respectively, with indices if needed. The transformations $\cdot A_1$, $\cdot B_1$, $A_2\cdot$ and $B_2\cdot$ are called *input*, *output*, *cost* and *revenue* transformations. When forming a von Neumann model we start with the intensity space R^n and select a cone S of those intensities which are applicable, then we pass to the space of prices $\tilde{R}^m$ and select a cone of possible prices T. The matrices A_1 and B_1 depict the technological possibilities of the system, and the matrices A_2 and B_2 — the economic processes of forming costs and revenues.

Let us emphasize that S and T are not supposed to be closed, but both are supposed to be pointed. The dual cones S^* and T^* are closed and solid as a consequence of the definition of the dual cone.

By the definition of a von Neumann model, all four spaces under consideration have natural bases and, consequently, natural coordinatewise ordering following those bases. The ordering by cones S, T^*, S^* and T can be different. However, in the economic considerations (but less in this paper) the connections between those two orders in each space are important. A reasonable assumption is, for instance, that $T \subset {}^+\tilde{R}^m$, i.e. that possible prices are non-negative.

3. Von Neumann models: classification. A von Neumann model in which $A_1 = A_2 = A$ and $B_1 = B_2 = B$ is called *simple* and can be written as

$$\mathfrak{M} = (R^n, S, \tilde{R}^m, T, A, B).$$

We refer to models which are not simple as to *extended models.* A special kind of extended von Neumann models are *three-matrix models* in which $A_1 = A_2$ and $B_1 \neq B_2$ or $A_1 \neq A_2$ and $B_1 = B_2$. These two forms of three-matrix models are mutually dual.

If the ordering cones S and T are closed, we call the model *closed* (it should not be confused with models of closed economies); if they are polyhedral, we call the model *polyhedral.* A polyhedral model is closed. If S and T are non-negative orthants of the corresponding spaces, i.e. $S = {}^+R^n$ and $T = {}^+\tilde{R}^m$, we call the model *normal.* An extended normal model can be regarded as a four-tuple of matrices, and when it is also simple, as a pair of matrices (A, B).

A model with all transformations monotone is called *monotone.* A normal model is monotone iff all its matrices are non-negative.

We will single out classes of von Neumann models with one or more of the following properties:

(KMT_1) There exists a $p \geqslant 0$ with $A_1 p > 0$.

(KMT_2) There exists an $x \geqslant 0$ with $xB_2 > 0$.

(Ł) For every $x \geqslant 0$ and $p \geqslant 0$ there exists a number $\zeta > 0$ such that $\zeta x A_1 p \leqslant x A_2 p$ and $\zeta x B_2 p \leqslant x B_1 p$.

(vN) For every $x \gtrapprox 0$ and every $p \gtrapprox 0$ at least one of the quantities $xA_1p, xA_2p, xB_1p, xB_2p$ is positive.

(BKŁ) (for simple models only) For every $x \geqslant 0$ and $\lambda > 0$, if $\lambda x A \leqslant xB$, then $xB \geqslant 0$.

Let us notice that the vector inequalities, as used above, refer to different ordered spaces. Due to the convention that vectors in R^n, R^m, $\tilde{R}^n$, $\tilde{R}^m$ are denoted by x, y, q, p, respectively, we can determine from the shape of letters and the character of transformations to which space the inequality refers.

The properties (KMT) are dual, i.e. if a model has one of them, then the dual model has the other. The properties (Ł) and (vN) are self-dual. The property (BKŁ) is not self-dual.

The first paper to investigate von Neumann models in ordered spaces is probably Łoś [17]. The assumptions on orderings have gradually been weakened to achieve finally the form presented here, where S and T are only assumed to be pointed but even not to be closed. A special form of extended models was introduced in the paper Morgenstern and Thompson [30]. The properties (KMT) have been introduced in the paper Kemeny et al. [14]. Originally expressed for normal, monotone, simple models they read: no row of A is zero and no column of B is zero. The property (vN) for the same class of models (von Neumann [33]) reads as follows: $A+B$ has all entries positive.

4. Equilibria of von Neumann models: definition and properties. For every von Neumann model $\mathfrak{M}$ we define a two-person, non-zero-sum, min-oriented game with strategies being semi-positive vectors $x \gtrapprox 0$ and $p \gtrapprox 0$ for the first and the second player, respectively, and the pay-off functions defined as follows:

$f(x, p) = xA_2p//xB_2p$ for the player controlling x's;

$g(x, p) = xB_1p//xA_1p$ for the player controlling p's.

The function of two real variables $\cdot//\cdot$ is defined as

$$a//\beta = \sup\{\lambda \mid \lambda\beta \leqslant a\}$$

or, equivalently,

$$a//\beta = \begin{cases} a/\beta & \text{iff } \beta > 0, \\ -\infty & \text{iff } \beta = 0 \text{ and } a < 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Since both pay-off functions are positively homogeneous of degree zero (i.e. $f(x, p) = f(\zeta x, \psi p)$ for any $\zeta > 0$ and $\psi > 0$) and both S and T are pointed cones, we can restrict the strategy sets to pre-compacts (i.e. compact after closure) taking any $q > 0$ and $y > 0$ and putting

$$Q = \{x \geqslant 0 \mid xq = 1\} \quad \text{and} \quad P = \{p \geqslant 0 \mid yp = 1\}.$$

The game played with strategies restricted to Q and P is in all respects equivalent to that played with $x \gtrapprox 0$ and $p \gtrapprox 0$.

Let us recall that, since both players are minimizing, the *non-cooperative (Nash's) equilibrium* of this game is a pair of strategies $\bar{x} \gtrapprox 0$ and $\bar{p} \gtrapprox 0$ such that

$$f(\bar{x}, \bar{p}) \leqslant f(x, \bar{p}) \quad \text{for all } x \gtrapprox 0,$$

$$g(\bar{x}, \bar{p}) \leqslant g(\bar{x}, p) \quad \text{for all } p \gtrapprox 0.$$

A non-cooperative equilibrium of the game defined above is called a *von Neumann equilibrium* of the corresponding model or, for shortness, an *equilibrium of the model.* At the equilibrium $\bar{x}$ and $\bar{p}$, the values of the pay-off functions $f(\bar{x}, \bar{p})$ and $g(\bar{x}, \bar{p})$ can be finite or infinite. The first case is more important, but we cannot dispose completely of the second one.

THEOREM 4A. *A pair of vectors $\bar{x} \gtrsim 0$ and $\bar{p} \gtrsim 0$ is an equilibrium with finite values iff there exist two numbers λ and σ such that*

(4.1) $\lambda \bar{x} A_1 \leqslant \bar{x} B_1,$ (4.2) $\sigma B_2 \bar{p} \leqslant A_2 \bar{p},$

(4.3) $\lambda \bar{x} A_1 \bar{p} = \bar{x} B_1 \bar{p},$ (4.4) $\sigma \bar{x} B_2 \bar{p} = \bar{x} A_2 \bar{p},$

(4.5) $\bar{x} A_1 \bar{p} > 0,$ (4.6) $\bar{x} B_2 \bar{p} > 0.$

(Then, of course, $\lambda = g(\bar{x}, \bar{p})$ and $\sigma = f(\bar{x}, \bar{p})$.) If $\lambda > 0$ and $\sigma > 0$, then, in economic terminology, λ is the *factor of growth* and $\mu = \sigma^{-1}$ is the *factor of interest.* The existence of those factors can be secured only in special cases.

THEOREM 4B. *If the model has both the properties* (KMT) *and the property* (Ł), *then all equilibria* (*if any*) *have finite, positive values and, consequently, $\bar{x} \gtrsim 0$ and $\bar{p} \gtrsim 0$ is an equilibrium iff there exist two positive numbers $\lambda > 0$ and $\sigma > 0$ such that* (4.1)-(4.6) *are satisfied.*

The definition of an equilibrium for a normal, monotone model with the property (vN) has originally been given by von Neumann [33] as a minimax-maximin pair of strategies of a zero-sum game. The definition above follows that idea of von Neumann, but since it concerns more general models, it uses Nash's equilibria in non-zero games. Theorems 4A and 4B can be found in Łoś [20].

5. Equilibria of extended models: existence. The only known theorem on existence of equilibria for a comparatively large class of extended von Neumann models is

THEOREM 5A. *If a closed and monotone model has both the properties* (KMT) *and the property* (Ł), *then there exist vectors $\bar{x} \gtrsim 0$ and $\bar{p} \gtrsim 0$ and two numbers λ, σ such that conditions* (4.1)-(4.4) *are satisfied. If, moreover, the model has the property* (vN), *then, for every $\bar{x} \gtrsim 0$, $\bar{p} \gtrsim 0$, λ, σ for which* (4.1)-(4.4) *hold,* (4.5) *and* (4.6) *also hold and $\lambda > 0$, $\sigma > 0$, thus the pair $\bar{x}$ and $\bar{p}$ is an equilibrium.*

The proof uses Kakutani's fixed-point theorem and goes very much along the lines of the von Neumann proof in his original paper [33]. Let us point out the necessity of the property (Ł); there exist monotone models with both the properties (KMT) and the property (vN) but deprived of equilibria.

Theorem 5A is a very weak one indeed. Not only the assumption of the property (vN) is restrictive. In most of interesting economic problems the models are not monotone. There are non-monotone models, deprived of all four properties assumed in Theorem 5A, which have a lot of equilibria with finite, even positive values (see Łoś and Łoś [23]).

Theorem 5A for three-matrix, normal models has been proved in Łoś [18]. In Ballarini and Moeschlin [1] this proof has been generalized to include some non-normal models. The proof for extended models with four matrices and arbitrary (closed) orderings is contained in Łoś [20].

6. Equilibria of extended models: equilibrium levels. By the *set of equilibrium levels* of a von Neumann model we understand the set of pairs of positive numbers (λ, μ) such that for λ and $\sigma = \mu^{-1}$ there exist vectors $\bar{x} \underset{\cap}{>} 0$ and $\bar{p} \underset{\cap}{>} 0$ satisfying (4.1)-(4.6). An equivalent definition states that (λ, μ) is *in the set of equilibrium levels* if $f(\bar{x}, \bar{p}) = \lambda > 0$ and $g(\bar{x}, \bar{p}) = \mu^{-1} > 0$ for some equilibrium $\bar{x}$ and $\bar{p}$.

The exact shape of the sets of equilibrium levels has not been known yet but the following can be proved:

THEOREM 6A. *The set of equilibrium levels of an extended von Neumann model is a union of a finite number of open rectangles with sides parallel to the axes which eventually can degenerate into intervals and points.*

It follows from this theorem that we have either a finite number of equilibrium levels or continuum of them.

Theorem 6A has not been published yet.

7. Aggregations and extensions of von Neumann models. Let us consider two von Neumann models

$$\mathfrak{M} = (R^n, S, \tilde{R}^m, T, A_1, B_1, A_2, B_2)$$

and

$$\mathfrak{M}' = (R^l, S', \tilde{R}^k, T', A_1', B_1', A_2', B_2')$$

and two monotone linear mappings (matrices of dimensions $l \times n$ and $m \times k$)

$$\cdot L\colon (R^l, S') \to (R^n, S) \quad \text{and} \quad K\cdot\colon (\tilde{R}^k, T') \to (\tilde{R}^m, T).$$

We have then two diagrams

$$\begin{array}{ccc}
(R^n, S) & \xrightarrow{\cdot A_1, \cdot B_1, \cdot A_2, \cdot B_2} & (R^m, T^*) \\
\uparrow {\scriptstyle \cdot L} & & \downarrow {\scriptstyle \cdot K} \\
(R^l, S') & \xrightarrow{\cdot A_1', \cdot B_1', \cdot A_2', \cdot B_2'} & (R^k, T'^*)
\end{array}$$

$$\begin{tikzcd}
(\tilde{R}^m, T) \arrow[rr, "A_1\cdot{,}B_1\cdot{,}A_2\cdot{,}B_2"] & & (\tilde{R}^n, S^*) \arrow[d, "L\cdot"] \\
(\tilde{R}^k, T') \arrow[u, "K\cdot"] \arrow[rr, "A_1'\cdot{,}B_1'\cdot{,}A_2'\cdot{,}B_2'\cdot"] & & (\tilde{R}^l, S'^*)
\end{tikzcd}$$

By definition, if both diagrams commute, i.e. $A_i' = LA_iK$ and $B_i' = LB_iK$ for $i = 1, 2$, then (alternatively)

if $\cdot L\colon S' \to S$ is an injection (one-to-one into) and $K\cdot\colon T' \to T$ is an injection, then (L, K) is called an *aggregation* of $\mathfrak{M}$ into $\mathfrak{M}'$, and $\mathfrak{M}'$ is an *aggregate* of $\mathfrak{M}$;

if $\cdot L\colon S' \to S$ and $K\cdot\colon T' \to T$ are surjections ($S'L = S$, $KT' = T$), then (L, K) is said to *extend* $\mathfrak{M}$ to $\mathfrak{M}'$, and $\mathfrak{M}'$ is called an *extension* of $\mathfrak{M}$.

It is easy to see that if S and T are solid (thus generating the whole space), then $k \leqslant m$ and $l \leqslant n$ imply the existence of an aggregation, whereas $n \leqslant l$ and $m \leqslant k$ imply the existence of an extension.

Let us suppose now that (L, K) is an aggregation of $\mathfrak{M}$ into $\mathfrak{M}'$ and that we have two additional linear monotone mappings

$$\cdot\bar{L}\colon (R^n, S) \to (R^l, S') \quad \text{and} \quad \bar{K}\cdot\colon (\tilde{R}^m, T) \to (\tilde{R}^k, T').$$

If $\cdot L\bar{L}\colon R^l \to R^l$ and $\cdot\bar{K}K\colon R^k \to R^k$ are both identities, then the quadruple $(L, \bar{L}, K, \bar{K})$ is called an *aggregaton*. An aggregaton yields the following diagram (the dual diagram is omitted):

$$\begin{tikzcd}
(R^n, S) \arrow[rr, "\cdot A_1{,}\cdot B_1{,}\cdot A_2{,}\cdot B_2"] \arrow[d, shift left, "\cdot\bar{L}"] & & (R^m, T^*) \arrow[d, shift left, "\cdot K"] \\
(R^l, S') \arrow[u, shift left, "\cdot L"] \arrow[rr, "\cdot A_1'{,}\cdot B_1'{,}\cdot A_2'{,}\cdot B_2'"] & & (R^k, T'^*) \arrow[u, shift left, "\cdot\bar{K}"]
\end{tikzcd}$$

Let M be some transformation from among A_1, B_1, A_2, B_2. Then we have (alternatively)

(I) If $\bar{L}M' = MK$, then the aggregaton is called *left-intrinsic for M*.

(II) If $M'\bar{K} = LM$, then the aggregaton is called *right-intrinsic for M*.

(III) If $M = \bar{L}M'\bar{K}$, then the aggregaton is called *perfect for M*.

If the aggregaton is perfect for all four transformations (i.e. $(\bar{L}, \bar{K})$ extends $\mathfrak{M}'$ to $\mathfrak{M}$), then it is called, simply, *perfect*, and $\mathfrak{M}'$ — a *perfect aggregate of* $\mathfrak{M}$.

THEOREM 7A. *Each polyhedral model is a perfect aggregate of a normal model.*

THEOREM 7B. *For a model $\mathfrak{M}$ in order to have an extension which is a closed model it is necessary and sufficient that both ordering cones S and T are F_σ (unions of a countable number of closed sets).*

THEOREM 7C. *Let $(L, \bar{L}, K, \bar{K})$ be an aggregaton of the model $\mathfrak{M}$ into the model $\mathfrak{M}'$. Then*

(a) *if $x \geqslant 0$ and $p \geqslant 0$ is an equilibrium of $\mathfrak{M}$ at the levels (λ, μ), then $x' \geqslant 0$ and $p' \geqslant 0$ is an equilibrium of $\mathfrak{M}'$ at the same levels if one of the following conditions holds:*

(i) *$x'L = x$ and $Kp' = p$,*

(ii) *the aggregaton is left-intrinsic for A_1, B_1 and $x' = x\bar{L}$, $Kp' = p$,*

(iii) *the aggregaton is right-intrinsic for A_2, B_2 and $x'L = x$, $p' = \bar{K}p$,*

(iv) *the aggregaton is left-intrinsic for A_1, B_1 and right-intrinsic for A_2, B_2, and $x' = x\bar{L}$, $p' = \bar{K}p$;*

(b) *if $x' \geqslant 0$ and $p' \geqslant 0$ is an equilibrium of $\mathfrak{M}'$ at the levels (λ, μ), then $x \geqslant 0$ and $p \geqslant 0$ is an equilibrium of $\mathfrak{M}$ at the same levels if one of the following conditions holds:*

(v) *the aggregaton is right-intrinsic for A_1, B_1 and left-intrinsic for A_2, B_2, and $x'L = x$, $Kp' = p$,*

(vi) *the aggregaton is perfect for A_1, B_1 and left-intrinsic for A_2, B_2, and $x' = x\bar{L}$, $Kp' = p$,*

(vii) *the aggregaton is right-intrinsic for A_1, B_1 and perfect for A_2, B_2, and $x'L = x$, $p' = \bar{K}p$,*

(viii) *the aggregaton is perfect for all transformations of the model and $x' = x\bar{L}$, $p' = \bar{K}p$.*

It follows that if $\mathfrak{M}'$ is a perfect aggregate of $\mathfrak{M}$, then for every (λ, μ) the mapping

$$(\cdot\bar{L}, K\cdot): R^n \times \tilde{R}^m \to R^l \times \tilde{R}^k$$

maps the set of equilibria at the levels (λ, μ) of $\mathfrak{M}$ onto the set of equilibria at the levels (λ, μ) of $\mathfrak{M}'$.

The aggregations of simple, normal and monotone models have been studied by the author in a series of lectures given at the University in Aarhus in 1967 (see Łoś [16]). The presentation has been based mainly on general ideas proposed by E. Malinvaud and A. Nataf. Since then some authors familiar with notes of the Aarhus lectures have applied aggregation to more general von Neumann models (e.g. Dang Quang [8], Le quang Hung [15], Sosnowska [36]). Aggregations, the more extensions of von Neumann models as presented in this section, have not been published yet. Theorem 7B is an easy consequence of the Bromek and Kaniewski theorem (see Section 1).

8. Quasi-equilibria of simple models. Since at the equilibrium of a simple von Neumann model both equilibrium levels, if finite, are equal and positive, then an equilibrium of such models can be defined as two vectors $\bar{x} \geqslant 0$ and $\bar{p} \geqslant 0$ for which there exists a positive number $\lambda > 0$ such that

$$(8.1) \qquad \lambda \bar{x} A \leqslant \bar{x} B, \qquad (8.2) \qquad B\bar{p} \leqslant \lambda A \bar{p}, \qquad (8.3) \qquad \bar{x} B \bar{p} > 0.$$

If the model is polyhedral and $\lambda > 0$, $\bar{x} \geqslant 0$ satisfy (8.1), then in order to solve (8.2) and (8.3) with $\bar{p} \geqslant 0$, so that $\bar{x}$ and $\bar{p}$ become an equilibrium at the level λ, it is necessary and sufficient that

$$(8.4) \qquad \bar{x}B \leqslant x(B - \lambda A) \text{ has no solution } x \geqslant 0.$$

If the model is not polyhedral, then (8.4) turns out to be a necessary but not a sufficient condition to solve (8.2) and (8.3).

A vector $\bar{x} \geqslant 0$ for which, with the given $\lambda > 0$, (8.1) and (8.4) hold is called a *quasi-equilibrium* of the corresponding model. The number λ, which is unique for $\bar{x}$, is called the *level* of that quasi-equilibrium.

Quasi-equilibria of polyhedral models can always be completed by a price vector to become equilibria. This is, however, not true for non-polyhedral models. It is easy to construct pure quasi-equilibria from equilibria of normal models by removing some faces of the non-negative orthant on which equilibrium price vectors lie. By taking a closed extension we obtain a closed model which has a pure quasi-equilibrium. There are known examples of closed models with quasi-equilibria but without equilibria which cannot be obtained in a similar way. The first example of a von Neumann model (a cone model, indeed, see Section 16) with only quasi-equilibria has been constructed in Movshovich [32]. Since then many other examples have been published (see, for instance, Bromek et al. [6], Section 14, p. 125-129).

9. The gross production function and the levels of quasi-equilibria. The multifunction $G\colon {}^{+}R \rightsquigarrow T^{*}$ defined as

$$G(\lambda) = \{y \geqslant 0 \mid \lambda x A \leqslant x B \geqslant y \text{ for some } x \geqslant 0\}$$

is called the *gross production function* (g.p.f.) of the simple model $\cdot A, \cdot B\colon (R^n, S) \to (R^m, T^*)$. For every $\lambda > 0$, the set $G(\lambda)$ is a *face* of the cone T^* (i.e. a sub-cone F such that if $y \in F$, $y' \in T^*$ and $y - y' \in T^*$, then $y' \in F$). If the model has the property (BKŁ), then its g.p.f. is non-increasing in the sense that $G(\lambda_2) \subset G(\lambda_1)$ for $\lambda_1 \leqslant \lambda_2$. From this section up to Section 15 we shall consider only models having the property (BKŁ).

Let us call the *character of the cone* T^* the maximal length r of a descending chain of different faces of the cone T^*:

$$T^* = F_0 \underset{\neq}{\supset} F_1 \underset{\neq}{\supset} \dots \underset{\neq}{\supset} F_r = \{0\}.$$

The values of a g.p.f., when λ increases, constitute a chain (which cannot be a maximal one) and, therefore, a g.p.f. is constant everywhere but in a finite number of points where it has jumps (down) and the number of those jumps cannot exceed the character of T^*.

At any point λ where a g.p.f. has a jump either

(9.1) there exists a $y \in G(\lambda)$ such that

$$y \notin G(\lambda+) = \bigcup_{\varepsilon>0} G(\lambda+\varepsilon), \quad \text{i.e.} \quad G(\lambda)\setminus G(\lambda+) \neq \emptyset,$$

(9.2) there exists a $y \notin G(\lambda)$ such that

$$y \in G(\lambda-) = \bigcap_{\varepsilon>0} G(\lambda-\varepsilon), \quad \text{i.e.} \quad G(\lambda-)\setminus G(\lambda) \neq \emptyset,$$

or both.

THEOREM 9A. *A sufficient and necessary condition for $\lambda > 0$ to be a quasi-equilibrium level is that, for the g.p.f. of the model,* (9.1) *holds at λ. All quasi-equilibria x at the level λ are such that $xB \in G(\lambda)\setminus G(\lambda+)$.*

THEOREM 9B. *If the model is polyhedral, then* (9.2) *implies* (9.1) *and, therefore, every jump of the g.p.f. is a level of the equilibrium.*

It follows from Theorem 9A that the number of quasi-equilibrium levels is bounded from above by the character of the cone T^*. Applying Theorem 9A to the dual model we infer that the number of equilibrium levels is also bounded by the character of the cone S^*. Since both characters do not surpass the dimensions of the spaces, then we have a rough boundary for the number of equilibrium levels $\min(n, m)$.

Quasi-equilibria and the g.p.f.'s have been introduced in the paper Bromek et al. [6], where the proofs of main results concerning those notions can be found. Theorem 9B, in the form given here, is proved in Bromek [4]. Recently, Berezneva [2] has reported on results very similar to those contained in this section. She investigates simple and normal models which cannot be monotone and characterizes equilibrium levels of such models very much along Theorem 9B. She uses also (independently) the property (BKŁ).

10. Existence of equilibria and quasi-equilibria. The properties (KMT_1) and (KMT_2) imply that for small but positive λ's the g.p.f. equals T^* and for λ's large enough it equals $\{0\}$. Therefore, the g.p.f. of a model having both properties (KMT) (and the property (BKŁ)) has jumps and, if the model is polyhedral, then all such jumps are equilibrium levels.

However, if the model is not polyhedral, then some of these jumps can be of type (9.2) but not of (9.1), thus they will not be even quasi-equilibrium levels. If the model is deprived of the property (KMT_2), it can happen (even if the model is closed) that no jumps are of type (9.1); thus the model has no quasi-equilibria at all. This cannot happen if the model is closed and has both properties (KMT).

THEOREM 10A. *A closed model with both properties* (KMT) *has always quasi-equilibria, and one of them is the maximal factor of growth*

$$\check{\lambda} = \inf\{\lambda \mid G(\lambda) = \{0\}\}.$$

The minimal factor of profit

$$\hat{\lambda} = \sup\{\lambda \mid G(\lambda) = T\}$$

cannot be a level of quasi-equilibrium, but since $\hat{\lambda}^{-1}$ is the maximal factor of growth of the dual model and the assumptions of Theorem 10A are self-dual, $\hat{\lambda}^{-1}$ is a quasi-equilibrium level of the dual model provided the dual model has the property (BKŁ).

THEOREM 10B. *In a polyhedral model with both properties* (KMT), *both $\check{\lambda}$ and $\hat{\lambda}$ are levels of equilibria* (*it may happen that $\check{\lambda} = \hat{\lambda}$*).

Since von Neumann [33] gave a proof of existence of an equilibrium and since Kemeny et al. [14] generalized his definition and gave a proof (Thompson [37]) of existence under milder and economically more meaningful assumptions, many new proofs of the existence have been published. An account of these researches can be found in the introductions of papers Bromek et al. [6] and Łoś [20]. Recently, in Morgenstern and Thompson [31] the paper by Kemeny [13] was published for the first time. It contains a proof of existence of equilibrium different from the others as applying perturbation arguments.

Let us mention here also the paper by Moeschlin [28] supplemented by his book [29]. By applying a theorem of Mills on derivatives of the value function $v(\lambda) = v(B-\lambda A)$ of the matrix game $B-\lambda A$, he proves indeed the following generalization of the existence theorem:

If the system of inequalities $x(B-\lambda A) \geqslant v(\lambda)e$, $(B-\lambda A)p \geqslant v(\lambda)f$, $xBp > 0$ has no solution in probability vectors x and p, then $v(\lambda)$ is constant in a neighbourhood of λ.

It follows that

$$\sup\{\lambda \mid v(\lambda) = 0\} = \check{\lambda} \quad \text{and} \quad \inf\{\lambda \mid v(\lambda) = 0\} = \hat{\lambda},$$

if finite and positive, are equilibrium levels. (Here $e = (1, 1, \ldots, 1)$ and $f = (1, 1, \ldots, 1)$.)

11. Equilibria and quasi-equilibria of Leontief models. A simple, monotone model with $(R^n, S^*) = (R^m, T^*)$ and I being the identity transformation, thus of the form

$$\cdot A, \cdot I: (R^n, S) \to (R^n, S),$$

is called a *Leontief model*. A Leontief model, indeed, reduces to one transformation $\cdot A: R^n \to R^n$ which preserves a closed, pointed and solid cone S: $SA \subset S$. The Perron-Frobenius theory investigates such situation in the normal case. No wonder therefore that the theory of quasi-equilibria of Leontief models generalizes in some respect the Perron-Frobenius theory.

THEOREM 11A. *If $\bar{x} \geqslant 0$ is an eigenvector with the eigenvalue $\mu > 0$ of the transformation $\cdot A$ in a Leontief model, then $\bar{x}$ is a quasi-equilibrium of that model at the level $\lambda = \mu^{-1}$.*

Obviously, not all quasi-equilibria of Leontief models are eigenvectors. This happens only at the highest possible level, i.e. for the maximal growth factor.

THEOREM 11B. *If $\check{\lambda}$ is the maximal growth factor of a Leontief model, then every $x \geqslant 0$ satisfying $\check{\lambda} xA \leqslant x$ is an eigenvector with the eigenvalue $\mu = \check{\lambda}^{-1}$.*

At other quasi-equilibrium levels we can have quasi-equilibria which are not eigenvectors but there is at least one which is.

THEOREM 11C. *If $\lambda > 0$ is a quasi-equilibrium level of a Leontief model, then $\mu = \lambda^{-1}$ is an eigenvalue of $\cdot A$ associated with a non-negative eigenvector.*

The existence of an eigenvector in a convex, closed and pointed cone for every transformation which maps this cone into itself and which does not take into zero any non-zero vector in that cone follows from Theorems 11A, 11B, 11C and 10A. However, it seems to be more interesting that from Theorem 9A it follows that there exists a boundary on the number of eigenvalues associated with vectors in a cone preserved by the linear transformation, and that this boundary depends only on the shape of the cone.

THEOREM 11D. *If the character of a closed and pointed cone S in R^n is r, then, for any linear transformation which maps S into itself, the number of eigenvalues associated with eigenvectors in S is less than or equal to r.*

For proofs of these theorems see Bromek et al. [6]. Theorem 11B for normal models has been proved in Gale [10].

12. Quasi-Leontief models. A model

$$\cdot A, \cdot B: (R^n, S) \to (R^m, T^*)$$

is called *quasi-Leontief* if

(12.1) for every $x' \geqslant 0$ the equality $x'A = x''B$ has a solution $x'' \geqslant 0$;

(12.2) for every $p' \geqslant 0$ the equality $Ap' = Bp''$ has a solution $p'' \geqslant 0$.

We have also a special kind of quasi-Leontief models, i.e. *almost-Leontief models*, which are of the form

$$\cdot A, \cdot I: (R^n, S) \to (R^n, T^*)$$

with $\cdot A$ preserving S and $A\cdot$ preserving T.

Quasi-Leontief and even almost Leontief models cannot be monotone. The results of the previous section can be extended to quasi-Leontief models, with some restrictions, however, imposed on cones and/or on transformations.

THEOREM 12A. *If a quasi-Leontief model is polyhedral, has the property* (BKŁ) *and T^* is pointed, then, for any its equilibrium $\bar{x} \geqslant 0$ and $\bar{p} \geqslant 0$ at the level λ, there exists an $\bar{\bar{x}} \geqslant 0$ such that $\lambda\bar{\bar{x}}A = \bar{\bar{x}}B$, and $\bar{\bar{x}}$ and $\bar{p}$ is an equilibrium at the same level λ.*

THEOREM 12B. *If a quasi-Leontief model is monotone and T^* is pointed, then every vector $\bar{x}$ such that $\lambda\bar{x}A = \bar{x}B \neq 0$ for some $\lambda > 0$ is a quasi-equilibrium at the level λ.*

THEOREM 12C. *Every polyhedral quasi-Leontief model is an extension of an almost Leontief model.*

For proofs of these theorems see Sosnowska [36].

13. Stability of von Neumann models. The problem is in showing how the properties of a model change when entries of its matrices are perturbed. A simple model is considered as a point in R^{2nm} and a norm in this space is used to indicate how far a perturbation is from the perturbed model.

We consider a simple model with matrices (A, B), a subset C in R^{2nm} to which (A, B) belongs and a feasible pair (x, λ), i.e. such that $x \geqslant 0$, $\lambda > 0$ and $\lambda xA \leqslant xB$. We say that (A, B) is *stable* at (x, λ) with respect to C if for some $\varepsilon > 0$ and $\beta > 0$ and every model $(\tilde{A}, \tilde{B})$ in C such that $\|(A, B) - (\tilde{A}, \tilde{B})\| < \varepsilon$ there exists a pair $(\tilde{x}, \tilde{\lambda})$ with $\tilde{x} \geqslant 0$, $\tilde{\lambda} > 0$ and $\tilde{\lambda}\tilde{x}\tilde{A} \leqslant \tilde{x}\tilde{B}$ such that

$$\max\{\|\tilde{x} - x\|, |\tilde{\lambda} - \lambda|\} \leqslant \beta \min\{\|y\| \mid x(\tilde{B} - \tilde{\lambda}\tilde{A}) + y \geqslant 0\}.$$

THEOREM 13A. *A normal and monotone model (A, B) with the property* (KMT_1) *is stable with respect to the non-negative orthant of R^{2nm} at a feasible (x, λ) if and only if every $p \gtrsim 0$ such that $Bp \leqslant \lambda Ap$ forms with x an equilibrium at the level λ.*

THEOREM 13B. *A normal and monotone model (A, B) with the property* (KMT_1) *is stable at every feasible (x, λ) with respect to the face $C(A, B)$ of the non-negative orthant of R^{2nm} on which (A, B) lies.*

Since $C(A, B) = \{(\tilde{A}, \tilde{B}) \geqslant 0 \mid a_{ij} = 0$ implies $\tilde{a}_{ij} = 0$ and $b_{ij} = 0$ implies $\tilde{b}_{ij} = 0$ for all i and $j\}$ we see that, by restricting perturbations to $C(A, B)$, we can perturb only positive entries in A and B.

Let now $\Lambda(A, B)$ denote the set of equilibrium levels of the normal and monotone model (A, B). Since $\Lambda(A, B)$ is a finite set, it can be regarded as a point in the metric space of compact subsets of real numbers with the Hausdorff distance

$$\operatorname{dist}(X, Y) = \max\,[\sup_{x\in X}\inf_{y\in Y}|x-y|,\ \sup_{y\in Y}\inf_{x\in X}|y-x|].$$

By this definition, $\Lambda(\cdot, \cdot)$ becomes a function from the space R^{2nm} to the metric space of compacts and, therefore, investigations of its continuity are meaningful.

THEOREM 13C. *For every normal and monotone model* (A, B), *the function* $\Lambda(\cdot, \cdot)$ *is continuous at* (A, B) *on* $C(A, B)$.

Theorem 13A is due to Robinson [35]. Theorems 13B and 13C are due to Kaniewska, the first one is unpublished, the second one is contained in [12].

14. Other multifunctions connected with simple models. The *net production function* (n.p.f.) is a multifunction $N\colon {}^+R \rightsquigarrow T$ defined as follows:

$$N(\lambda) = \{y \geqslant 0 \mid x(B-\lambda A) \geqslant y \text{ for some } x \geqslant 0\}.$$

The *function of possible intensities* (f.p.i.) is a multifunction $I\colon {}^+R \rightsquigarrow S$ defined as follows:

$$I(\lambda) = \{x \geqslant 0 \mid \lambda x'A \leqslant x'B \text{ and } x \leqslant x' \text{ for some } x' \geqslant 0\}.$$

Both n.p.f. and f.p.i. have properties similar to those of the g.p.f. They are non-increasing and their values are faces of the corresponding cones, of T for n.p.f. and of S for f.p.i. There are, however, some important differences between the behaviour of those functions and of the g.p.f., which are to be seen from the following theorems:

THEOREM 14A. *Let the input transformation* $\cdot A$ *be monotone. Then*

(a) $N(\lambda) \neq G(\lambda)$ *iff* λ *is a level of quasi-equilibrium*;

(b) *if the model is polyhedral, then* λ *is a level of equilibrium iff* (9.2) *holds at* λ *for* N (*instead of* G), *i.e. if* $N(\lambda-)\setminus N(\lambda) \neq \emptyset$.

THEOREM 14B. *If the output transformation* $\cdot B$ *is monotone, then* $G(\cdot)$ *has jumps at those and only those points* λ *where* $I(\cdot)$ *has jumps.*

Let us note that it is easy to construct a normal model with the properties (BKŁ) and both (KMT) but with non-monotone $\cdot B$ such that, at some λ being a level of equilibrium, $I(\cdot)$ does not have a jump.

These results have not been published yet. They are due mainly to Hoang phong Oanh and Bromek.

15. Algorithm for finding equilibrium levels. A finite set $Z \subset R$ is *ε-located* if there are given a finite number of intervals of the total length less than $\varepsilon > 0$, such that their union covers the set Z and in every interval there lies at least one element of Z. There is a standard procedure to locate jumps of any g.p.f., n.p.f. or f.p.i. provided we know how to compute the values of that function or, at least, if we have an algorithm to decide if for two numbers the function takes different values and if, moreover, we can find two numbers λ_1 and λ_2 such that the function has no jumps for $\lambda < \lambda_1$ and $\lambda > \lambda_2$. If the latter is satisfied, we have a rough location with $\varepsilon = \lambda_2 - \lambda_1$ to start with. Going step by step we can refine every location by splitting all intervals into two and rejecting those at the beginning and at the end of which the function takes the same values.

Such an algorithm can obviously be constructed for normal models with the property (BKŁ) using the g.p.f. and, by Theorem 9B, it will locate all levels of equilibria of the model. In the case where, moreover, $\cdot A$ is monotone, the n.p.f. can be used, and if $\cdot B$ is monotone, also the f.p.i. can be used.

For normal and monotone models the algorithm using essentially the f.p.i. was proposed by Thompson [38] and Morgenstern and Thompson [31], it seems, however, that the justification of the algorithm offered by Thompson is not correct.

16. Production cones and von Neumann models. For a simple von Neumann model

$$\cdot A, \cdot B: (R^n, S) \to (R^m, T^*),$$

the *production cone* is defined as follows:

$$P(\cdot A, \cdot B) = \{(xA, xB) \mid x \geqslant 0\}.$$

If the model is closed and has the property (KMT$_1$), then $P(\cdot A, \cdot B)$ is a closed and pointed cone. If the model is monotone, then $P(\cdot A, \cdot B) \subset T^* \times T^*$, in particular for a normal and monotone model the production cone lies in the non-negative orthant $^+R^m \times {}^+R^m$ of R^{2m}.

In many problems concerning von Neumann models we can restrict ourselves to consider only the production cone $P = P(\cdot A, \cdot B) \subset R^{2m}$ and the cone T in the space of prices $\tilde{R}^m$. Doing so we get rid of the original transformation model and we deal with the von Neumann cone model, called also the *von Neumann - Gale model*, since Gale [9], [11] has used it extensively.

By a *von Neumann cone model* we mean a space (of price vectors) $\tilde{R}^m$ ordered by a pointed cone T and a pointed cone $P \subset R^{2m}$. An *equilibrium* of a cone model at the level $\lambda > 0$ is defined as a pair $(\bar{y}_1, \bar{y}_2) \in P$ and a price vector $\bar{p} \in T$ such that

(16.1) $\lambda \bar{y}_1 \leqslant \bar{y}_2$ (the inequality refers to T_1^*);

(16.2) for every $(y_1, y_2) \in P$ we have $y_2\bar{p} \leqslant \lambda y_1 \bar{p}$;

(16.3) $\bar{y}_2\bar{p} > 0$.

A *representation* of the cone model is every transformation model $\cdot A, \cdot B: (R^n, S) \to (R^m, T^*)$ such that $P(\cdot A, \cdot B) = P$. It can be shown that every cone model has many representations and one of them is the following. As the intensity space we take R^{2m} ordered by the cone P, as the space of price vectors we take $\tilde{R}^m$ ordered by T and we define $\cdot A, \cdot B: (R^{2m}, P) \to \to (R^m, T^*)$ as projections $(y_1, y_2)A = y_1$ and $(y_1, y_2)B = y_2$. This transformation model is called *standard* for the original cone model. It is obviously a representation and is a minimal one in the sense of the following

THEOREM 16A. *Every representation of a cone model is an extension of its standard representation.*

The cone model, although handy in several dynamic problems, loses one important economic aspect: it has no dual. With any transformation model beside the production cone $P(\cdot A, \cdot B)$ another production cone, this of the dual model, is related:

$$P = P(B\cdot, A\cdot) = \{\langle Bp, Ap\rangle \mid p \geqslant 0\}.$$

This cone lies in the space $\tilde{R}^{2n}$, and R^n is ordered by S, thus it produces another cone model. If, however, two models are the same cone models, i.e. $P(\cdot A, \cdot B) = P(\cdot \bar{A}, \cdot \bar{B})$, then, in general, the production cones of their dual models $P(B\cdot, A\cdot)$ and $P(\bar{B}\cdot, \bar{A}\cdot)$ can be quite different. They usually lie in spaces of different dimensions.

Theorem 16A and more details about converting cone models into transformation models are contained in the paper by Le quang Hung [15].

REFERENCES

[1] C. Ballarini and O. Moeschlin, *An open von Neumann model with consumption*, in [26], p. 1-10.

[2] T. D. Berezneva, *The structure of equilibrium levels in von Neumann type models*, Abstract of a lecture on IX International Symposium on Mathematical Programming, Budapest, August 23-27, 1976.

[3] T. Bromek, *Consumption-investment frontier in von Neumann models*, in [24], p. 47-58.

[4] — *Contributions to the theory of existence of von Neumann equilibria (II)*, in [26], p. 11-21.

[5] — J. Kaniewska and J. Łoś, *Equilibria of von Neumann models and eigenvectors of monotone transformations, Announcement of results*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 22 (1971), p. 707-709.

[6] — *Contributions to the theory of existence of von Neumann equilibria*, in [25], p. 103-131.

[7] T. Bromek and J. Kaniewski, *Linear images and sums of closed cones in Euclidean spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 24 (1976), p. 231-238.

[8] Dang Quang, *Regional models of a closed economy*, in [25], p. 203-216.

[9] D. Gale, *The closed linear model of production*, in: *Linear inequalities and related systems*, Princeton 1956, p. 385-403.

[10] — *The theory of linear economic models*, New York 1960.

[11] — *On optimal development in a multi-sector economy*, Review of Economic Studies 34 (1967), p. 1-18.

[12] J. Kaniewska, *Stability of equilibria of an expanding economy*, in: Third Reisenburg Symposium *On the stability of contemporary economic systems*, Reisenburg 1977 (to appear).

[13] J. G. Kemeny, *Game-theoretic solution of an economic problem*, in [31], Appendix 4, p. 251-255.

[14] — O. Morgenstern and G. L. Thompson, *A generalization of the von Neumann model of an expanding economy*, Econometrica 24 (1956), p. 115-135.

[15] Le quang Hung, *Von Neumann models defined by transformations and by production cones*, in [26], p. 49-66.

[16] J. Łoś, *Linear methods in the theory of economic models*, Lecture Notes, Aarhus University 1967.

[17] — *A simple proof of the existence of equilibrium in a von Neumann model and some of its consequences*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 19 (1971), p. 971-979.

[18] — *Labour, consumption and wages in a von Neumann model*, in [24], p. 67-72.

[19] — *The existence of equilibrium in an open expanding economy model (Generalization of the Morgenstern-Thompson model)*, in [24], p. 73-80.

[20] — *Extended von Neumann models and game theory*, in [25], p. 141-157.

[21] — *Von Neumann models of open economics*, in [26], p. 67-96.

[22] — and M. W. Łoś, *Remarks on efficiency frontiers in von Neumann models*, in [25], p. 159-169.

[23] — *Reswitching of techniques and equilibria of extended von Neumann models*, in [26], p. 97-118.

[24] — (editors), *Mathematical models in economics*, Amsterdam - New York - Warszawa 1974.

[25] — (editors), *Computing equilibria: How and why*, Amsterdam - New York - Warszawa 1976.

[26] — and A. Wieczorek (editors), *Warsaw fall seminars in mathematical economics 1975*, Lecture Notes in Economics and Mathematical Systems 133, Berlin - Heidelberg - New York 1976.

[27] L. Mardoń, *The Morgenstern-Thompson model of an open economy in a closed form*, in [24], p. 81-114.

[28] O. Moeschlin, *Derivatives of game value functions in connection with von Neumann growth models*, in [24], p. 115-125.

[29] — *Zur Theorie von Neumannscher Wachstumsmodelle*, Lecture Notes in Economics and Mathematical Systems, Berlin - Heidelberg - New York 1974.

[30] O. Morgenstern and G. L. Thompson, *Private and public consumption and savings in the von Neumann model of an expanding economy*, Kyklos 20 (1967), p. 387-409.

[31] — *Mathematical theory of expanding and contracting economics*, Lexington 1976.

[32] S. M. Movshovich, *Theorems on a turnpike in a von Neumann-Gale models* (in Russian), Ekonomiya i matematičeskiye metody 5 (1969), p. 877-889.

[33] J. von Neumann, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines mathematischen Kolloquiums 8 (1937), p. 73-83.

[34] — *A model of general economic equilibrium*, Review of Economic Studies 12 (1945-1946), p. 1-9.

[35] S. M. Robinson, *Policy and price stability in the von Neumann model of a closed economy*, in [25], p. 171-179.

[36] H. Sosnowska, *Quasi-Leontief models*, in [26], p. 119-130.

[37] G. L. Thompson, *On the solution of a game-theoretic problem*, in: *Linear inequalities and related systems*, Princeton 1956.

[38] — *Computing the natural factors of a closed expanding economy*, Zeitschrift für Nationalökonomie 34 (1974), p. 57-68.

COMPUTING CENTRE
OF THE POLISH ACADEMY OF SCIENCES
WARSZAWA

Reçu par la Rédaction le 21. 1. 1977