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UC-SETS IN THE DUAL OBJECT OF COMPACT GROUPS

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Introduction. UC-sets were firstly defined by Figà-Talamanca [4] on the one-dimensional torus and studied in detail by Pedemonte [8] who extended this notion to general compact Abelian groups. For a study of properties of UC-sets in the Abelian case we refer also to [3], [2] and [11].

In this paper we attempt to extend the notion of UC-set to non-Abelian compact groups. In Section 1 we give the definition and prove some characterizations which generalize well-known results; as a consequence of such characterizations we are able to show that any compact group contains UC-sets in great profusion and that UC-sets and central UC-sets coincide. In Section 2 we study UC-sets over SU_2, giving a concrete example and showing that the union problem has a negative answer in the non-Abelian case.

1. Let G be a compact group with a countable dual object $\hat{G}$ and let X_n be finite subsets of $\hat{G}$ such that

$$X_0 = \emptyset, \tag{1}$$

$$X_n \subset X_{n+1} \quad \text{for all } n, \tag{2}$$

$$\bigcup_{n=0}^{\infty} X_n = \hat{G}. \tag{3}$$

If, for each $\sigma \in \hat{G}$, $\hat{f}(\sigma)$ is the unique operator on the finite-dimensional Hilbert space H_σ such that

$$\langle \hat{f}(\sigma)\xi, \eta\rangle = \int_G f(x)\langle U_{x^{-1}}^{(\sigma)}\xi, \eta\rangle dx \quad \text{for all } \xi, \eta \in H_\sigma$$

and

$$f \sim \sum_{\sigma\in\hat{G}} d_\sigma \operatorname{tr}\big(\hat{f}(\sigma)\, U^{(\sigma)}\big),$$

we put

$$S_n f(x) = \sum_{\sigma\in X_n} d_\sigma \operatorname{tr}\big(\hat{f}(\sigma)\, U_x^{(\sigma)}\big).$$

Let E be a subset of $\hat{G}$.

Definition. E is called a *set of uniform convergence* (*UC-set*) if, whenever $f \in C_E(G)$ (where $C_E(G)$ is the space of continuous functions f on G with $\hat{f}(\sigma) = 0$ for every $\sigma \notin E$),

$$\|S_n f - f\|_u \to 0.$$

Remark 1. Obviously, the definition of UC-set depends on the choice of the sequence $\{X_n\}$. It is clear that a Sidon set is a UC-set for every choice of $\{X_n\}$, since

$$\|S_n f\|_u \leqslant \sum_{\sigma \in X_n} d_\sigma \operatorname{tr} |\hat{f}(\sigma)| \leqslant \delta \|f\|_u,$$

where δ is the Sidon constant of E (see Theorem 1 (ii) in the sequel).

Definition. We denote by $BV(E, \{X_n\})$ the space of the functions $\varphi \colon E \to C$ such that

$$\varphi(\sigma) = \text{const} = \varphi_n \quad \text{whenever } \sigma \in X_n \setminus X_{n-1},$$

$$\|\varphi\|_{BV} = \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| + \lim_{j\to\infty} |\varphi_j| < \infty.$$

When no confusion can arise, we will write BV for $BV(E, \{X_n\})$.

It is easy to prove that the function $\|\cdot\|_{BV}$ is a norm and BV is a Banach space with this norm. We denote by BV_0 the closed subspace of BV of all functions φ such that $\varphi_j \to 0$.

Pedemonte [8] proves a characterization of UC-sets for compact Abelian groups. Now we consider this problem when G is not necessarily Abelian and the following theorem improves Pedemonte's result.

THEOREM 1. *The following statements are equivalent:*

(i) *E is a UC-set.*

(ii) *There exists a constant c such that*

$$\|S_n f\|_u \leqslant c \|f\|_u \quad \textit{for all } f \in C_E(G) \textit{ and for all } n.$$

(iii) *There exists a constant c such that*

$$\sup_n \Big| \sum_{\sigma \in X_n} d_\sigma \operatorname{tr}\big(\hat{p}(\sigma)\big) \Big| \leqslant c \|p\|_u$$

for all trigonometric polynomials $p \in T_E(G)$ and for all n.

(iv) *There exists a constant c such that, whenever $\varphi \in BV(E, \{X_n\})$, we can find a measure $\mu \in M(G)$ such that*

$$\hat{\mu}(\sigma) = \varphi(\sigma) I_\sigma \quad \textit{for all } \sigma \in E, \tag{4}$$

$$\|\mu\|_1 \leqslant c \|\varphi\|_{BV}. \tag{5}$$

(v) *There exists a constant k such that, whenever $\varphi \in BV_0(E, \{X_n\})$, we can find a function $g \in L^1(G)$ such that*

$$\hat{g}(\sigma) = \varphi(\sigma) I_\sigma \text{ for all } \sigma \in E, \qquad \|g\|_1 \leqslant k \|\varphi\|_{BV}.$$

(vi) *There exist a constant k and a sequence $\{g_n\} \subset L^1(G)$ such that*

$$\hat{g}_n(\sigma) = I_\sigma \quad \text{whenever } \sigma \in E \cap X_n, \tag{6}$$

$$\hat{g}_n(\sigma) = 0 \quad \text{whenever } \sigma \in E \setminus X_n, \tag{7}$$

$$\|g_n\|_1 \leqslant k \quad \text{for all } n. \tag{8}$$

Proof. We will prove the theorem by establishing the following implications: (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii) is obvious by the uniform boundedness principle.

(ii) $\Rightarrow$ (i). For each $\varepsilon > 0$ there exists a trigonometric polynomial p such that $\|p\|_1 = 1$ and $\|f*p - f\|_u < \varepsilon$. Then, if $X_n \supset \operatorname{supp}(\hat{p})$, we have

$$\|S_n f - f\|_u \leqslant \|S_n f - f*p\|_u + \|f*p - f\|_u \leqslant (c+1)\varepsilon.$$

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). For $\varphi \in BV$ let

$$T(p) = \sum_{\sigma \in \hat{G}} d_\sigma \varphi(\sigma) \operatorname{tr}\big(\hat{p}(\sigma)\big) \quad \text{for all } p \in T_E(G).$$

For n sufficiently large we have $X_n \supset \operatorname{supp}(\hat{p})$, and so

$$\begin{aligned} T(p) &= \sum_{j=1}^{n} \varphi_j \sum_{\sigma \in X_j \setminus X_{j-1}} d_\sigma \operatorname{tr}\big(\hat{p}(\sigma)\big) \\ &= \sum_{j=1}^{n-1} (\varphi_j - \varphi_{j+1}) \sum_{\sigma \in X_j} d_\sigma \operatorname{tr}\big(\hat{p}(\sigma)\big) + \varphi_n \sum_{\sigma \in X_n} d_\sigma \operatorname{tr}\big(\hat{p}(\sigma)\big). \end{aligned}$$

Hence we have

$$|T(p)| \leqslant \|\varphi\|_{BV} \sup_n \Big| \sum_{\sigma \in X_n} d_\sigma \operatorname{tr}\big(\hat{p}(\sigma)\big) \Big| \leqslant c \|\varphi\|_{BV} \|p\|_u.$$

By the Hahn-Banach theorem, the linear functional T can be extended to a linear functional on $C(G)$, also denoted by T, where $\|T\| \leqslant c\|\varphi\|_{BV}$. By the Riesz representation theorem, there exists a measure $\nu \in M(G)$ such that

$$\|\nu\|_1 \leqslant c \|\varphi\|_{BV} \quad \text{and} \quad \int_G f d\nu = T(f) \text{ for all } f \in C(G).$$

If $\tau \in E$, let $\{e_i\}$ be an orthonormal base on H_τ. Then, by the orthogonality relations $(u_{ij}^{(\tau)}(x) = \langle U_x^{(\tau)} e_j, e_i\rangle)$,

$$\langle\hat{\nu}(\bar{\tau})e_i, e_j\rangle = \int_G \langle U_{x^{-1}}^{(\bar{\tau})} e_i, e_j\rangle d\nu = \sum_{\sigma\in\hat{G}} \varphi(\sigma) d_\sigma \operatorname{tr}\big((u_{ij}^{(\tau)})^\wedge(\sigma)\big) = \varphi(\tau)\delta_{ij}.$$

Hence $\hat{\nu}(\bar{\tau}) = \varphi(\tau)I_\tau$, and this shows that the measure μ with $\mu(A) = \nu(A^{-1})$ satisfies (4) and (5).

(iv) $\Rightarrow$ (v). Without loss of generality, let $\varphi \in BV_0$, where $\|\varphi\|_{BV} = 1$. We construct a sequence of non-negative integers in the following way: let $n_0 = 0$ and $n_k > n_{k-1}$ such that $|\varphi_n| < (\frac{1}{2})^k$ for all $n \geqslant n_k$. For every function

$$\varphi^{(k)}(\sigma) = \begin{cases} \varphi(\sigma) & \text{if } \sigma \in X_{n_k} \setminus X_{n_{k-1}}, \\ 0 & \text{elsewhere} \end{cases}$$

there exists a measure μ_k which satisfies (4) and (5). Let

$$a_k = \sum_{j=n_k+1}^{n_{k+1}} |\varphi_{j+1} - \varphi_j|;$$

then

$$\|\mu_{k+1}\|_1 \leqslant c\|\varphi^{(k+1)}\|_{BV} = c(|\varphi_{n_{k+1}}| + |\varphi_{n_k+1}| + a_k) \leqslant c[(\tfrac{1}{2})^{k-1} + a_k].$$

Moreover, there exists $h_k \in L^1(G)$, where $\hat{h}_k(\sigma) = I_\sigma$ for $\sigma \in X_{n_k} \setminus X_{n_{k-1}}$ and $\|h_k\|_1 \leqslant 2$, and so

$$\|h_{k+1} * \mu_{k+1}\|_1 \leqslant 2c[(\tfrac{1}{2})^{k-1} + a_k].$$

By the Lebesgue theorem, we have

$$g = \sum_{k=1}^{\infty} h_k * \mu_k \in L^1(G) \quad \text{and} \quad \|g\|_1 \leqslant 6c.$$

(v) $\Rightarrow$ (vi) follows obviously from (v), where

$$\varphi^{(n)}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in X_n, \\ 0 & \text{elsewhere}. \end{cases}$$

(vi) $\Rightarrow$ (ii). If g_n satisfies (6)-(8), we have

$$\|S_n f\|_u = \|f * g_n\|_u \leqslant k\|f\|_u \quad \text{for all } f \in C_E(G).$$

The characterization above shows that there is no difference between UC-sets and central UC-sets (i.e., the sets $E \subset \hat{G}$ such that $\|S_n f - f\|_u \to 0$ whenever $f \in C_E(G)$ is central). Actually, we have the following result:

THEOREM 2. *$E \subset \hat{G}$ is a UC-set iff $\|S_n f - f\|_u \to 0$ for every central function $f \in C_E(G)$.*

Proof. For every central trigonometric polynomial $p \in T_E(G)$ we put

$$T(p) = \sum_{\sigma\in\hat{G}} d_\sigma \varphi(\sigma)\operatorname{tr}(\hat{p}(\sigma)).$$

It is easy to prove that $|T(p)| \leqslant c\|\varphi\|_{BV}\|p\|_u$ (see Theorem 1). We now use the Hahn-Banach theorem and the fact that the dual of $C^z(G)$ is $M^z(G)$, where $C^z(G)$ and $M^z(G)$ denote the centers of the algebras $C(G)$ and $M(G)$ with respect to convolution (see [7]). Therefrom we deduce that there exists a central measure ν such that

$$\|\nu\|_1 \leqslant c\|\varphi\|_{BV} \quad \text{and} \quad \int_G f d\nu = T(f)$$

for every central function $f \in C_E(G)$. Then

$$\operatorname{tr}(\hat{\nu}(\bar{\tau})) = \sum_{\sigma\in\hat{G}} \varphi(\sigma) d_\sigma \operatorname{tr}(\chi_\tau)^\wedge(\sigma) = d_\tau \varphi(\tau).$$

The measure ν is central and, therefore, $\hat{\nu}(\bar{\tau}) = \varphi(\tau)I_\tau$; the assertion follows at once from Theorem 1 (iv).

Remark 2. By Theorem 2 every central Sidon set is a UC-set for every choice of $\{X_n\}$. Moreover, it is easy to prove that the converse is true.

The next theorem shows that, unlike other kinds of lacunary sets, UC-sets always exist in compact non-Abelian groups.

THEOREM 3. *Let $\{X_n\}$ be any sequence of $\hat{G}$ which satisfies* (1)-(3). *Then every infinite sequence $\{\sigma_k\} \subset \hat{G}$ contains an infinite UC-set (with respect to the sequence $\{X_n\}$).*

Proof. Without loss of generality, we suppose that $\sigma_k \in X_{n_k}$ for all k, where $X_{n_1} \subset X_{n_2} \subset \ldots \subset X_{n_k} \subset \ldots$ Let $\sigma_{k_1} = \sigma_1$; there exists a central trigonometric polynomial h_1 such that $\|h_1\|_1 = 1$ and $\hat{h}_1(\sigma) = c_\sigma^{(1)} I_\sigma$ for every $\sigma \in \hat{G}$, where $|c_{\sigma_1}^{(1)} - 1| < d_{\sigma_1}^{-2}$. We inductively select $\sigma_{k_2}, \sigma_{k_3}, \ldots, \sigma_{k_j}, \ldots$ in $\{\sigma_k\}$ and the trigonometric polynomials $h_2, h_3, \ldots, h_j, \ldots$ in $T(G)$ so that $\hat{h}_{j-1}(\sigma_h) = 0$ for all $h \geqslant k_j$ and $\|h_j\|_1 = 1$ and $\hat{h}_j(\sigma) = c_\sigma^{(j)} I_\sigma$ for all $\sigma \in \hat{G}$, where $|c_{\sigma_{k_i}}^{(j)} - 1| < d_{\sigma_{k_i}}^{-2}/j$ $(i = 1, 2, \ldots, j)$. We prove that $E = \{\sigma_{k_j}\}$ is a UC-set by establishing (vi) of Theorem 1. For this purpose we consider the functions

$$p_j = h_j + \sum_{i=1}^{j} (1 - c_{\sigma_{k_i}}^{(j)}) d_{\sigma_{k_i}} \chi_{\sigma_{k_i}}.$$

We estimate

$$\Big\|\sum_{i=1}^{j} (1 - c_{\sigma_{k_i}}) d_{\sigma_{k_i}} \chi_{\sigma_{k_i}}\Big\|_1 \leqslant \sum_{i=1}^{j} d_{\sigma_{k_i}} |1 - c_{\sigma_{k_i}}^{(j)}| \int_G |\chi_{\sigma_{k_i}}(x)|\, dx \leqslant 1,$$

and so $\|p_j\|_1 \leqslant 2$.

Letting $g_n(x) = p_j(x)$ whenever $n_{k_j} \leqslant n < n_{k_{j+1}}$, we have

$$\hat{g}_n(\sigma) = \begin{cases} I_\sigma & \text{if } \sigma \in E \cap X_n, \\ 0 & \text{if } \sigma \in E \setminus X_n, \end{cases}$$

and the assertion follows from Theorem 1 (vi).

Remark 3. The definition of UC-set makes sense when the whole of $\hat{G}$ is not a UC-set with respect to the sequence $\{X_n\}$. Suppose, e.g., that G is the Cantor group and X_n is the subgroup of $\hat{G}$ of all elements with coordinates $x_j = 0$ if $j \geqslant n+1$. Then $\hat{G}$ is a UC-set with respect to $\{X_n\}$. However, if we order in any way the elements of $\hat{G}$, say, $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ and let $X_n = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, then, G being Abelian, $\hat{G}$ is not a UC-set. It is not known whether a similar result holds true for general non-Abelian compact groups (**P 1170**). However, this is the case for SU_2 (see [9]).

Remark 4. Pedemonte [8] proved that every Abelian group G contains a UC-set which is not Sidon. We were not able to prove the analogous result for the non-Abelian case (**P 1171**). However, we remark that a large class of non-Abelian groups does not admit central Sidon sets (see [10]).

2. Throughout this section G is the group SU_2; we denote by σ_l the class of the continuous irreducible unitary representations on SU_2 of dimension $2l+1$ for all non-negative half-integer l (see [5], p. 125), and choose the sequence $\{X_n\}$ as follows:

$$X_0 = \emptyset, \quad X_1 = \{\sigma_0\}, \quad \ldots, \quad X_n = \{\sigma_0, \sigma_{1/2}, \ldots, \sigma_{(n-1)/2}\}, \ldots$$

In the Abelian case it is not known whether the union of two UC-sets is a UC-set (¹); for SU_2 we have a negative answer.

THEOREM 4. *There exist two UC-sets whose union is not a UC-set.*

The proof of this theorem uses the following lemma (see also [6]):

LEMMA. *Let p be a positive integer and let $E \subset \hat{G}$ be a UC-set; then E cannot contain infinitely many pairs σ_k, σ_{k+p}.*

Proof. We suppose that E contains an infinite number of such pairs $\sigma_{k_i}, \sigma_{k_i+p}$ and without loss of generality we can assume that

$$\sum_{i=1}^{\infty} d_{k_i}^{-1} = \sum_{i=1}^{\infty} (2k_i+1)^{-1} < \infty \quad \text{and} \quad k_i + p < k_{i+1}.$$

We consider the trigonometric polynomials

$$p_m(x) = \sum_{i=1}^{m} (2k_i+1)^{-1} [\chi_{k_i+p}(x) - \chi_{k_i}(x)].$$

(¹) As J. J. Fournier (Vancouver) has recently shown, the answer is in general negative [Note of the editors].

If x_t is the general element of the maximal torus of SU_2, then

$$p_m(x_t) = \begin{cases} 2\,\dfrac{\sin(\frac{1}{2}pt)}{\sin(\frac{1}{2}t)} \displaystyle\sum_{i=1}^{m} (2k_i+1)^{-1}\cos\big(k_i+\tfrac{1}{2}(p+1)\big)t & \text{if } t \in (0, 2\pi), \\ 2p \displaystyle\sum_{i=1}^{m} (2k_i+1)^{-1} & \text{if } t = 0. \end{cases}$$

The sequence $\{p_m\}$ converges uniformly on G to a continuous function f, so $S_n p_m \to S_n f$ for $n = 1, 2, \ldots$ However, $S_n f$ cannot converge uniformly to f, since

$$\|S_{2k_i+1} f - S_{2k_i} f\|_u = \|(2k_i+1)^{-1}\chi_{k_i}\|_u = 1.$$

Proof of Theorem 4. Let $\{\sigma_k\}$ be a sequence of $\hat{G}$ and p a positive integer. By Theorem 3, there exists a sequence of integers $\{k_i\}$ such that both $\{\sigma_{k_i}\}$ and $\{\sigma_{k_i+p}\}$ are UC-sets; but by the previous lemma the set $E = \{\sigma_{k_i}\} \cup \{\sigma_{k_i+p}\}$ is not a UC-set.

We want now to construct an explicit example of a UC-set on SU_2. This example can be produced by a direct application of the arguments used in the proof of Theorem 3. First of all we need a "nice" summability kernel; we may consider the following one constructed by Clerc [1]:

$$s_{2l+1}^{\delta}(x) = \sum_{i=0}^{\infty} (2i+1)\Phi_\delta\left(\frac{2i+1}{2l+1}\right)\chi_i(x),$$

where

$$\Phi_\delta(t) = \begin{cases} (1-t)^\delta & \text{if } 0 \leqslant t \leqslant 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Clerc proved that

$$\int_G |s_{2l+1}^{\delta}(x)|\,dx < C$$

uniformly with respect to l when $\delta > 1$. Obviously,

$$\hat{s}_{2l+1}^{\delta}(\sigma_i) = \begin{cases} \left(1 - \dfrac{2i+1}{2l+1}\right)^\delta I_{2i+1} & \text{if } i < l, \\ 0 & \text{if } i \geqslant l. \end{cases}$$

Suppose now that $\sigma_{l_1}, \sigma_{l_2}, \ldots, \sigma_{l_k}$ have been chosen. Arguing as in Theorem 3, we find the relation between l_{k+1} and l_k, i.e.,

$$1 - \left(1 - \frac{2l_k+1}{2l_{k+1}+1}\right)^\delta < \frac{(2l_k+1)^{-2}}{k},$$

so that if the sequence l_k satisfies

$$2l_{k+1} > \delta k (2l_k+1)^3,$$

then $\{\sigma_{l_k}\}$ is a UC-set.

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