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COMPATIBLE FUNCTION SEMIGROUPS

BY

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A *function* on a set A is any (in particular, empty) partial transformation of A. Sometimes functions are called *transformations*. If φ and ψ are functions on A, then $\psi\circ\varphi$ denotes the result of *superposition* of the pair (φ, ψ), i.e. $\psi\circ\varphi(a) = \psi(\varphi(a))$ for all $a \in A$ for which the right-hand side of the equality is defined (the left-hand side is defined simultaneously with the right-hand side). Thus $\psi\circ\varphi$ is just the composite function. A *function semigroup* is any non-empty set of functions (on a fixed set A) closed under superposition. The operation of superposition is associative, and so function semigroups are particular cases of semigroups. On the other hand, according to well-known Suschkewitsch's theorem, each abstract semigroup is isomorphic to a function semigroup. Therefore, the abstract theory of semigroups can be considered as an algebraic apparatus fit for studying the object of such a paramount mathematical importance as the function semigroup.

Functions φ and ψ on A are called *compatible* if $\varphi(a) = \psi(a)$ for all $a \in A$ for which both sides of this equality are defined (i.e., if the binary relation $\varphi\cup\psi$ is one-valued, that is, if $\varphi\cup\psi$ is a function). A set of functions is called *compatible* if every two functions from this set are compatible.

Univalent (i.e., one-to-one) functions φ and ψ on A are called *strictly compatible* if $\varphi\cup\psi$ is a univalent function (i.e., if φ, ψ as well as the inverse functions φ^{-1}, ψ^{-1} are compatible). A set of univalent functions is called *strictly compatible* if every two functions from this set are strictly compatible.

As a rule, compatible sets of functions are not closed under superposition, and sets of functions which are closed under superposition are not compatible. A (*strictly*) *compatible* (*univalent*) *function semigroup* is a (univalent) function semigroup with (strictly) compatible set of elements (which are functions). A semigroup isomorphic with a (strictly) compatible (univalent) function semigroup is called a (*strictly*) *compatible semigroup*.

As mentioned above, compatible semigroups form a rather narrow subclass of the class of semigroups. The aim of this paper is a description of the classes of compatible and strictly compatible semigroups. Our main results are the following two theorems:

THEOREM 1. *A semigroup is compatible if and only if it satisfies the identities*

(1) $$xy^2 = xy,$$

(2) $$xyz = xzy.$$

THEOREM 2. *A semigroup is strictly compatible if and only if it satisfies the identities*

(3) $$xy = (xy)^2,$$

(4) $$xy = yx.$$

Thus, both classes of semigroups are semigroup varieties, i.e., these classes are closed under subsemigroups, homomorphic images and direct products of arbitrary families of semigroups. See the end of this paper for a further discussion of the theorems and some results on the structure of (strictly) compatible semigroups (1).

Remark. A univalent mapping P of an abstract semigroup G onto a function semigroup is called an *isomorphism* if $P(gh) = P(h)\circ P(g)$ for all $g, h\in G$. The form of this equality is due to the fact that in the product gh the factors are read from left to right (thus, g is the first factor), however, in the product of functions $P(h)\circ P(g)$ the factors are read from right to left (and $P(g)$ is the first factor since it acts first in the composite function $P(h)\circ P(g)$, when this function acts on an element $a\in A$). If one prefers the equality $P(gh) = P(g)\circ P(h)$, one must change identities (1) and (2) for their dual (and read $y^2x = yx$ instead of (1)). Identities (3) and (4) are self-dual.

Proof of Theorem 1. Necessity. Let G be a compatible semigroup and let P be an isomorphism of G onto a compatible semigroup of functions on a set A. Let $g, h\in G$ and $a, b\in A$.

If $P(gh^2)(a) = b$, then $P(gh)(a)$ is defined since

$$P(gh^2)(a) = P(h)\big(P(gh)(a)\big).$$

Since $P(gh^2)$ and $P(gh)$ are compatible, $P(gh)(a) = P(gh^2)(a)$.

Suppose now $P(gh)(a)$ is defined. Since $P(gh)(a) = P(h)\big(P(g)(a)\big)$, we infer that $P(g)(a)$ is defined. Since $P(gh)$ and $P(g)$ are compatible functions, $P(gh)(a) = P(g)(a)$. It follows that

$$P(gh)(a) = P(h)\big(P(g)(a)\big) = P(h)\big(P(gh)(a)\big) = P(gh^2)(a).$$

(1) As J. Anusiak and K. Głazek have observed, the conjunction of (3) and (4) is equivalent to that of (1) and (4). (Note of the Editors.)

Thus, the functions $P(gh^2)$ and $P(gh)$ coincide. Since P is an isomorphism, $gh^2 = gh$ and (1) holds.

To verify (2) suppose $P(fgh)(a)$ is defined for some $f \in G$.

$$P(fgh)(a) = P(h)\big(P(fg)(a)\big) = P(gh)\big(P(f)(a)\big)$$

implies $P(f)(a)$ and $P(fg)(a)$ are defined. Since the functions $P(f), P(fg)$ and $P(fgh)$ are compatible,

$$P(f)(a) = P(fg)(a) = P(g)\big(P(f)(a)\big) = P(fgh)(a) = P(h)\big(P(fg)(a)\big)$$
$$= P(h)\big(P(f)(a)\big) = P(fh)(a).$$

Therefore,

$$P(fgh)(a) = P(fg)(a) = P(g)\big(P(f)(a)\big) = P(g)\big(P(fh)(a)\big) = P(fhg)(a).$$

In the same way one may prove that if $P(fhg)(a)$ is defined, then $P(fhg)(a) = P(fgh)(a)$. Therefore, the functions $P(fgh)$ and $P(fhg)$ coincide which implies $fgh = fhg$, and (2) holds.

Sufficiency. Suppose G is a semigroup satisfying (1) and (2). We are going to construct an isomorphism P of G onto a compatible semigroup of functions over an appropriate set A.

Let a_x and b_x be symbols corresponding to every $x \in G$; if $x, y \in G$ and $x \neq y$, then suppose all four symbols a_x, b_x, a_y and b_y are distinct. The set of all symbols a_x, b_x for all $x \in G$ is denoted by A. Let $P(g)$ be a function over A defined for every $g \in G$ as follows:

$P(g)(a_x)$ is defined iff $g = x$ or $gh = x$ for some $h \in G$;

if $P(g)(a_x)$ is defined, then $P(g)(a_x) = b_x$;

$P(g)(b_x)$ is defined iff $xg = x$, and this being the case $P(g)(b_x) = b_x$.

Clearly, $P(g)$ is a function, and the set of all functions $P(g)$ for $g \in G$ is compatible.

Suppose $P(g) = P(h)$. Evidently, $P(g)(a_g) = b_g$. Therefore, $P(h)(a_g)$ is defined, i.e., $g = h$ or $hx = g$ for some $x \in G$. Interchanging the roles of g and h, we obtain either $h = g$ or $gy = h$ for some $y \in G$. Suppose $hx = g$ and $gy = h$. By (1), $g = hx = hx^2 = gx$ which implies, by (2), $g = hx = gyx = gxy = gy = h$. Therefore, P is a univalent mapping of G onto a set of functions.

Let $P(h) \circ P(g)(a_x)$ be defined. Then $P(h) \circ P(g)(a_x) = b_x$ and both $P(g)(a_x)$ and $P(h)(b_x)$ are defined, i.e., $g = x$ or $gy = x$ for some $y \in G$, and $xh = x$. If $g = x$, then $gh = x$, and if $gy = x$, then $ghy = gyh = xh = x$. In both cases $P(gh)(a_x)$ is defined which implies $P(gh)(a_x) = b_x$. On the other hand, if $P(gh)(a_x)$ is defined, then $gh = x$ or $ghy = x$ for some $y \in G$. In both cases $P(g)(a_x)$ is defined. In the first case, $xh = gh^2 = gh = x$; in the second case, $xh = ghyh = gh^2y = ghy = x$, i.e. $P(h)(b_x)$ is defined. Therefore, $P(h) \circ P(g)(a_x)$ is defined and equals b_x.

Suppose now $P(h)\circ P(g)(b_x)$ is defined, i.e., both $P(g)(b_x)$ and $P(h)(b_x)$ are defined. In other words, $xg = x$ and $xh = x$. It follows that $xgh = xh = x$. If $xgh = x$, then

$$xg = xghg = xg^2h = xgh = x \quad \text{and} \quad xh = xgh = x.$$

The equality $xgh = x$ means that $P(gh)(b_x)$ is defined. Thus the domains of the functions $P(h)\circ P(g)$ and $P(gh)$ coincide. The definition of $P(g)$ and $P(h)$ and the fact just proved imply

$$P(h)\circ P(g) = P(gh).$$

Thus P is an isomorphism of G onto a compatible function semigroup, and G is compatible.

Proof of Theorem 2. Necessity. Let G be a strictly compatible semigroup. Then there exists an isomorphism P of G onto a strictly compatible semigroup of univalent functions. Define a new multiplication $*$ on the set of all elements of G: $g*h = hg$. Under this new operation G is a semigroup which will be denoted by G^*. G^* is called an *inverted semigroup* for G. Write $Q(g) = [P(g)]^{-1}$ for all $g \in G$. One can verify straightforwardly that Q is an isomorphism of G^* onto a strictly compatible semigroup of univalent functions. It follows that G and G^* are compatible semigroups. By Theorem 1, G and G^* satisfy (1) and (2). Clearly, G^* satisfies (1) and (2) iff G satisfies the identities $y^2x = yx$ and $zyx = yzx$. Using these identities as well as (1) and (2), we obtain

$$xy = xyy = yxy = yyx = yx.$$

Thus, G is commutative. Therefore,

$$xy = xy^2 = x^2y^2 = (xy)^2.$$

Hence identities (3) and (4) hold.

Sufficiency. Let G be a semigroup satisfying identities (3) and (4). Then identities (1) and (2) are also valid in G. In fact, (2) is a direct consequence of (4), while (1) follows from (3) and (4) in the following way:

$$xy^2 = (xy^2)^2 = x^2y^4 = x^2y^2 = (xy)^2 = xy.$$

As in the proof of Theorem 1, associate two symbols a_x and b_x with each $x \in G$. However, now we suppose $a_x = b_x$ iff $x^2 = x$. $P(g)$ is constructed precisely as in the previous case. One can easily verify that $P(g)$ is a function, i.e., every element in the domain of $P(g)$ has a uniquely determined image under $P(g)$. Clearly, the set of all functions $P(g)$ for $g \in G$ is compatible. Only the elements a_x and b_x can have the same image under $P(g)$. Let

$$P(g)(a_x) = P(g)(b_x).$$

Since the left-hand side of this equality is defined, $g = x$ or $gy = x$ for some $y \in G$. Since the right-hand side is defined, $xg = x$. If $g = x$, then $x^2 = x$. If $gy = x$, then

$$x^2 = xgy = xyg = gy^2g = gyg = xg = x.$$

In both cases $x^2 = x$, i.e., $a_x = b_x$. Therefore, the functions $P(g)$ are univalent for all $g \in G$. Suppose now $[P(g)]^{-1}(b_x)$ and $[P(h)]^{-1}(b_x)$ are both defined. Then the elements $[P(g)]^{-1}(b_x)$ and $[P(h)]^{-1}(b_x)$ are either equal or one of the elements equals a_x and the other element equals b_x. Without loss of generality we may suppose $[P(g)]^{-1}(b_x) = a_x$ and $[P(h)]^{-1}(b_x) = b_x$. Then $P(g)(a_x)$ and $P(h)(b_x)$ are both defined. Therefore, $g = x$ or $gy = x$ for some $y \in G$, and $xh = x$. By (3), $x^2 = (xh)^2 = xh = x$, i.e., $a_x = b_x$. Thus the set of functions $P(g)$ for all $g \in G$ is strictly compatible.

We can prove that $P(g) = P(h)$ implies $g = h$ exactly in the same way as in the proof of Theorem 1. It remains to verify the identity

$$P(gh) = P(h) \circ P(g) \quad \text{for all } g, h \in G.$$

The definition of the functions $P(g)$ and $P(h)$ entails the need to check the coincidence of the domains of $P(gh)$ and $P(h) \circ P(g)$, i.e., to check that $P(gh)(a_x)$ and $P(h) \circ P(g)(a_x)$ are defined or undefined simultaneously, and the same should be done for $P(gh)(b_x)$ and $P(h) \circ P(g)(b_x)$. In the case $a_x \neq b_x$ the verification of this fact coincides with the corresponding argument in the proof of Theorem 1. Suppose then $a_x = b_x$, i.e., $x^2 = x$. If $P(g)(a_x)$ is defined, then $g = x$ or $gy = x$ for some $y \in G$. In the first case $xg = x^2 = x$, in the second case $xg = gyg = g^2y = gy = x$, i.e., $P(g)(b_x)$ is defined. Conversely, if $P(g)(b_x)$ is defined, i.e., if $xg = x$, then $gx = xg = x$ and $P(g)(a_x)$ is defined. Therefore, $P(g)(a_x)$ and $P(g)(b_x)$ are defined or undefined simultaneously. Taking this into consideration, we may verify the needed fact in case $a_x = b_x$ exactly as it was done in the proof of Theorem 1.

Thus P is an isomorphism of G onto a strictly compatible semigroup of univalent functions.

Notice that Theorem 1 and the proof of the necessity in Theorem 2 imply that *a semigroup G is strictly compatible iff G and G^* are both compatible which, in turn, is equivalent to G being both isomorphic and anti-isomorphic to compatible function semigroups and this, finally, is equivalent to G being a commutative compatible semigroup.*

Let G be a semigroup and $(A_g)_{g \in G}$ a family of pairwise disjoint sets indexed by the elements of G. Suppose $g \in A_g$ for all $g \in G$. Introduce a binary operation $*$ on the set $A = \bigcup (A_g)_{g \in G}$: if $a \in A_g$ and $b \in A_h$, then $a * b = gh$. Then A with $*$ is a semigroup which is called an *inflation* of G [1].

A *semilattice* is an idempotent and commutative semigroup, i.e., a semigroup satisfying the identities $x^2 = x$ and $xy = yx$.

COROLLARY 1. *A semigroup is strictly compatible iff it is an inflation of a semilattice.*

Proof. Necessity. The verification of the fact that inflations of semilattices satisfy identities (3) and (4) is straightforward.

Sufficiency. Let G be a strictly compatible semigroup. By Theorem 2, G satisfies (3) and (4). Let φ denote the self-mapping of G defined as follows: $\varphi(a) = a^2$. By (4), φ is an endomorphism of G. Clearly, $\varphi(g) = g$ if g is an idempotent. For every $g \in G$, by (3), the element g^2 is idempotent. The set I of all idempotents is a subsemigroup of G since I is closed under multiplication. I is commutative and, therefore, I is a semilattice. Let A_i denote the set of all $g \in G$ such that $g^2 = i \in I$. Clearly, $(A_i)_{i \in I}$ is a partition of G and $i \in A_i$ for all $i \in I$. Let $g \in A_i$, $h \in A_j$. Then $gh = (gh)^2 = g^2h^2 = ij$, i.e., G is an inflation of the semilattice I. φ is a homomorphism of G onto I.

Since the inflation is a very simple semigroup-theoretic construction, the problem of the construction of all strictly compatible semigroups is reduced to the problem of constructing all semilattices.

Identities (1) and (2) which characterize compatible semigroups can be obtained from the identities of idempotence and commutativity if the latter identities are multiplied on the left by a new variable.

Every idempotent semigroup satisfies (1), therefore, idempotent compatible semigroups are those which satisfy (2), i.e., idempotent compatible semigroups are precisely restrictive semigroups of the second kind (left normal bands, in other terminology). The structure of such semigroups is known fairly well (see [2] and [5]).

If a non-empty set A is endowed with a binary operation such that $ab = a$ for all $a, b \in A$, A turns out to be a semigroup. Such semigroups are called *left zero semigroups*. One can easily verify that inflations of left zero semigroups satisfy (1) and (2), hence, they are compatible semigroups. The class of all such inflations is, in fact, characterized by two independent identities: the associativity and the identity $xy = x^2$.

Let ε be a congruence relation over a semigroup G and let all classes modulo ε be subsemigroups of G (in other words, the quotient semigroup G/ε is idempotent). This being the case, G is called a *band of the subsemigroups* G/ε (here G/ε denotes the set of all ε-classes and not the quotient semigroup). If the quotient semigroup is commutative (it satisfies identity (2)), we say that G is a *semilattice* (a *restrictive band of the second kind*) *of the subsemigroups* G/ε.

Inflations of one-element semigroups are called *zero semigroups* (thus, G is a zero semigroup iff $gh = 0$ for all $g, h \in G$; here 0 is the zero element of G).

COROLLARY 2. *Every compatible semigroup is a restrictive band of the second kind constructed from zero semigroups, and it is a semilattice of inflations of left zero semigroups.*

Proof. Let G be a compatible semigroup and let φ be a self-mapping of G: $\varphi(g) = g^2$ for all $g \in G$. If I is the set of all idempotents of G, then φ maps G onto I, since $\varphi(i) = i$ for $i \in I$ and $\varphi(g) = g^2 \in I$, by (1). The equalities

$$\varphi(gh) = (gh)^2 = ghgh = g^2h^2 = \varphi(g)\varphi(h)$$

show that φ is an endomorphism of G. Therefore, I is a subsemigroup of G. Let ε_φ denote the kernel congruence of φ. Since G satisfies (2), the quotient semigroup G/ε_φ satisfies (2) as well. Since I is idempotent, we infer that G/ε_φ is a decomposition of G into a restrictive band of the second kind. If g and h are elements of some ε_φ-class, then $gh = gh^2 = gg^2 = g^2 = \varphi(g) = \varphi(h)$, i.e., every class modulo ε_φ is a zero semigroup.

To prove the second part of Corollary 2 consider the following binary relation ε over G:

$$\varepsilon = \{(g, h):\ g^2 = g^2h,\ h^2 = h^2g\}.$$

By (1), ε is reflexive. Obviously, ε is symmetric. If $(f, g) \in \varepsilon$ and $(g, h) \in \varepsilon$, then

$$f^2 = f^2g = f^2g^2 = f^2g^2h = f^2gh = f^2hg = f^2h^2g = f^2h^2 = f^2h.$$

In the same way we can prove that $h^2 = h^2f$, i.e., $(f, h) \in \varepsilon$ and ε is transitive. Now, let $(f, g) \in \varepsilon$ and $(h, k) \in \varepsilon$. Then

$$(fh)^2 = f^2h^2 = f^2gh^2k = f^2h^2gk = (fh)^2(gk).$$

In the same way,

$$(gk)^2 = (gk)^2(fh).$$

Thus $(fh, gk) \in \varepsilon$ and ε is stable. Therefore, ε is a congruence relation over G. Clearly, $(g, g^2) \in \varepsilon$ for all $g \in G$, thus every ε-class is a subsemigroup of G. The equalities

$$(gh)^2 = g^2h^2 = g^3h^3 = g^2h^2hg = (gh)^2(hg) \quad \text{and} \quad (hg)^2 = (hg)^2(gh)$$

(the latter is a sequel of the former) show that $(gh, hg) \in \varepsilon$ for all $g, h \in G$. Thus G/ε is a semilattice, i.e., G is a semilattice of ε-classes.

It remains to prove that each ε-class is an inflation of a left zero semigroup. In effect, let H be an ε-class containing an element $g \in G$. Consider the self-mapping φ of H: $\varphi(h) = h^2$. Precisely as in the proof of the first part of Corollary 2, we can see that φ is an endomorphism of H onto an idempotent subsemigroup C. By the definition of φ, idempotence of C and definition of H, C is a left zero semigroup. It remains to note that $fh = fh^2 = fh^2f = \varphi(f)\varphi(h)$ for all $f, h \in H$, and thus H is an inflation of C.

If G is an abstract semigroup and P is an isomorphic representation of G by functions, one may introduce a binary relation ξ on G: $(g, h) \in \xi$ iff the functions $P(g)$ and $P(h)$ are compatible. Clearly, ξ depends on P. ξ is called a *compatibility relation.*

The following problem is of interest: what is an inner characterization of compatibility relations, what properties must a binary relation over G have in order it be a compatibility relation? This problem was solved in [7]. Compatible semigroups are precisely those for which the universal binary relation $G \times G$ is a compatibility relation. Thus, Theorem 1 could be deduced from the results of [7] (this was actually done), however, in this paper we gave a proof which does not depend on [7].

Another (still unsolved) problem is of interest: what are those semigroups for which the only compatibility relation is the identity relation? In other words, characterize semigroups G having the following property: if P is any isomorphic representation of G by functions and $g, h \in G$, $g \neq h$, then $P(g)$ and $P(h)$ are not compatible (**P 850**).

Main results of this paper are connected with the results of another paper [4], where the semigroups, which are isomorphic to function semigroups linearly ordered by the inclusion relation, are characterized. The trend of the theory of function and transformation semigroups which the present paper pertains to is the so-called *relation algebras* (see [3], [6] and [8]).

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