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BANACH SPACES OF COMPACT OPERATORS

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1. Introduction. In this paper (¹) a study is made of some Banach spaces of compact operators on a Hilbert space. The main result is obtained by using an abstract Lax-Milgram lemma to find the conjugate spaces of the reflexive spaces. These include the C_p-spaces ($1 < p < \infty$) and the other reflexive crossnorm spaces introduced by Schatten [5]. The Lax-Milgram technique avoids the use of many properties of Hilbert space which have been used in other methods.

Throughout the rest of the paper, H will denote a Hilbert space, $\mathscr{B}(H)$ the set of bounded linear operators on H, and C the set of compact operators in $\mathscr{B}(H)$. The usual operator norm for elements of $\mathscr{B}(H)$ will be denoted by $\|\cdot\|$, while $|T|$ will be used for $(T^*T)^{1/2}$.

Properties of the trace of an operator and the trace class can be found in [2] or [5]. In this paper the trace of an operator T will be denoted by $\operatorname{tr} T$.

2. C_ϱ-spaces. In this section we define some normed linear spaces of compact operators and prove some elementary properties.

Definition 2.1. A *generalized operator norm* ϱ is a function from the space of compact operators C to the extended real number system satisfying:

(i) $\varrho(T) \geqslant 0$ and $\varrho(T) = 0$ if and only if $T = 0$.

(ii) $\varrho(S+T) \leqslant \varrho(S)+\varrho(T)$ for all $S, T \in C$.

(iii) $\varrho(aT) = |a|\varrho(T)$ for all finite constants a.

In addition ϱ may satisfy the following:

(iv) $\varrho(T) < \infty$ for all T of rank 1.

(v) $\varrho(T) \geqslant \|T\|$ for all $T \in C$.

The *normed linear space* C_ϱ will be the space consisting of all $T \in C$ such that $\varrho(T) < \infty$. A *conjugate norm* ϱ' is defined on C in the following way:

(¹) The results in this paper are contained in the author's doctoral dissertation written under the direction of T. L. Hayden at the University of Kentucky.

Definition 2.2. For each $S \epsilon C$, define

$$\varrho'(S) = \sup_{\varrho(T)\leqslant 1} |\operatorname{tr} ST|,$$

where $\operatorname{tr} ST = \infty$ if ST does not belong to the trace class.

The following propositions are easily proved using elementary properties of the trace:

PROPOSITION 2.1. *If $C_{\varrho'}$ is the set of operators S such that $\varrho'(S) < \infty$ then $C_{\varrho'}$ is a linear space.*

PROPOSITION 2.2. *If ϱ satisfies property* (iv), *then ϱ' is a norm.*

The following gives a sufficient condition for C_ϱ to be complete:

THEOREM 2.1. *Suppose ϱ satisfies property* (v). *If*

$$\varrho\Big(\sum_{n=1}^{\infty} T_n\Big) \leqslant \sum_{n=1}^{\infty} \varrho(T_n)$$

for every sequence $\{T_n\}$ contained in C such that $\sum_{n=1}^{\infty} \varrho(T_n) < \infty$, then C_ϱ is complete.

Proof. Suppose

$$\sum_{n=1}^{\infty} \varrho(T_n) < \infty$$

for a sequence $\{T_n\}$ contained in C. Since ϱ satisfies property (v),

$$\sum_{n=1}^{\infty} \|T_n\| \leqslant \sum_{n=1}^{\infty} \varrho(T_n) < \infty$$

and hence $\sum_{n=1}^{\infty} T_n$ defines a compact operator.

Let $\{T_n\}$ be a sequence in C_ϱ such that $\varrho(T_n - T_m) \to 0$ as $n, m \to \infty$. There is a subsequence $\{S_n\} \subset \{T_n\}$ such that

$$\sum_{n=0}^{\infty} \varrho(S_{n+1} - S_n) < \infty,$$

where $S_0 = 0$. Let $U_n = S_{n+1} - S_n$, then U_n is compact and $\sum_0^{\infty} \varrho(U_n) < \infty$. By hypothesis,

$$\varrho\Big(\sum_0^{\infty} U_n\Big) \leqslant \sum_0^{\infty} \varrho(U_n)$$

and thus $\sum_0^{\infty} U_n$ belongs to C_ϱ. For each integer N,

$$\sum_{j=0}^{N} U_j = S_{N+1}$$

and thus

$$\varrho\Big(\sum_{0}^{\infty} U_n - S_{n+1}\Big) = \varrho\Big(\sum_{N+1}^{\infty} U_j\Big) \leqslant \sum_{N+1}^{\infty} \varrho(U_j)$$

which goes to 0 as $N \to \infty$. Now, for each positive integer n,

$$\varrho\Big(\sum_{0}^{\infty} U_n - T_n\Big) \leqslant \varrho\Big(\sum_{0}^{\infty} U_n - S_m\Big) + \varrho(S_m - T_n) \quad \text{for all } m.$$

Both terms tend to 0 as $n, m \to \infty$ and thus $\varrho\big(\sum_{0}^{\infty} U_n - T_n\big)$ tends to 0 as $n \to \infty$. Since $\sum_{0}^{\infty} U_n$ is in C_ϱ, it follows that C_ϱ is complete.

COROLLARY 2.1. *If $1 \leqslant p \leqslant \infty$, then C_p is complete.*

COROLLARY 2.2. *If ϱ' satisfies property* (v) *and*

$$\sup_{\varrho(T)\leqslant 1} |\operatorname{tr} ST| = \sup_{\varrho(T)\leqslant 1} \operatorname{tr}|ST| \quad \textit{for all } S \in C_{\varrho'},$$

then $C_{\varrho'}$ is complete.

Proof. Using a result of McCarthy's (see [4], lemma 3.1), we have

$$\Big|\operatorname{tr} \sum TT_n\Big| \leqslant \operatorname{tr}\Big|\sum TT_n\Big| \leqslant \liminf_{N\to\infty}\Big|\sum_{1}^{N} TT_n\Big| \leqslant \lim_{N\to\infty} \sum_{1}^{N} \operatorname{tr}|TT_n|.$$

PROPOSITION 2.3. *If $\varrho''(T) = \sup_{\varrho'(S)\leqslant 1} |\operatorname{tr} ST|$, then $\varrho''(T) \leqslant \varrho(T)$ for all $T \in C$.*

Proof. If $\varrho(T) = \infty$, then nothing is required. Suppose $\varrho(T) < \infty$ and $\varrho'(S) \leqslant 1$, then ST belongs to the trace class and $|\operatorname{tr} ST| \leqslant \varrho(T)$. Therefore $\varrho''(T) \leqslant \varrho(T)$.

THEOREM 2.2. *If ϱ satisfies property* (v), *then every element in the trace class belongs to $C_{\varrho'}$. In particular, every element of finite rank belongs to $C_{\varrho'}$ and hence ϱ'' is a norm.*

Proof. Let S be an element in the trace class. Then, for any $T \in \mathscr{B}(H)$, $|\operatorname{tr} ST| \leqslant \|T\| \operatorname{tr}|S|$. Since ϱ satisfies property (v), it follows that $|\operatorname{tr} ST| \leqslant \varrho(T) \operatorname{tr}|S|$ for all $T \in C_\varrho$. Thus

$$\sup_{\varrho(T)\leqslant 1} |\operatorname{tr} ST| \leqslant \operatorname{tr}|S| < \infty$$

and, therefore, S belongs to $C_{\varrho'}$. The last part follows from proposition 2.2.

3. Representations of linear functionals. The particular form of the abstract Lax-Milgram lemma which is used below is the one stated and proved by Hayden [3].

LEMMA 3.1 (Abstract Lax-Milgram). *Suppose U and V are Banach spaces, V is reflexive and that G is a bounded non-degenerate bilinear functional* *on $U \times V$. Then a necessary and sufficient condition that every bounded linear functional F on V have a unique representation of the form $F(v) = G(u, v)$ for a fixed $u \in U$ is that there exists an $m > 0$ such that*

$$\sup_{\|v\|=1} |G(u, v)| \geqslant m\|u\| \quad \textit{for each } u \in U.$$

We now apply this lemma to obtain the following representation theorem:

THEOREM 3.1. *Suppose C_ϱ and $C_{\varrho'}$ are Banach spaces, ϱ'' is a norm on C_ϱ, and C_ϱ is reflexive. Then, for each bounded linear functional F on C_ϱ, there is a unique S in $C_{\varrho'}$ such that $F(T) = \operatorname{tr} ST$ for all T in C_ϱ. Furthermore, $\|F\| = \varrho'(S)$.*

Proof. Define G from $C_{\varrho'} \times C_\varrho$ into the complex numbers by $G(S, T) = \operatorname{tr} ST$. Properties of the trace imply that G is bilinear and, from the definition of ϱ', we see that $|G(S, T)| \leqslant \varrho'(S)\varrho(T)$.

Suppose $G(S, T) = 0$ for all S in $C_{\varrho'}$ and a fixed $T \in C_\varrho$. Then $\operatorname{tr} ST = 0$ for all S in $C_{\varrho'}$ which implies that $\varrho''(T) = 0$. Since ϱ'' is a norm, this implies $T = 0$ and, therefore, G is non-degenerate.

Since

$$\sup_{\varrho(T)\leqslant 1} |G(S, T)| \geqslant \varrho'(S),$$

the theorem follows from the abstract Lax-Milgram lemma.

Using the results of McCarthy [4] that the C_p spaces $(1 < p < \infty)$ are uniformly convex and that S is in C_q $(1/p + 1/q = 1)$ if and only if ST belongs to the trace class for all T from C_p, we obtain his representation theorem.

COROLLARY 3.1. *If $1 < p < \infty$ and F is a bounded linear functional on C_p, then there is a unique S in C_q $(1/p + 1/q = 1)$ such that $F(T) = \operatorname{tr} ST$ for all $T \in C_p$. The norm of F is the q-norm of S.*

The results of Schatten [5] for reflexive crossnorm spaces also follow from theorem 2.2.

COROLLARY 3.2. *Suppose C_ϱ and $C_{\varrho'}$ are Banach spaces, C_ϱ is reflexive, and that ϱ and ϱ' are crossnorms. Then, for each bounded linear functional F on C_ϱ, there is a unique S in $C_{\varrho'}$ so that $F(T) = \operatorname{tr} ST$ for all T in C_ϱ and $\|F\| = \varrho'(S)$.*

THEOREM 3.2. *Suppose C_ϱ and $C_{\varrho'}$ are Banach spaces, C_ϱ is reflexive and ϱ'' is a norm. Then $\varrho(T) = \varrho''(T)$ for all $T \in C_\varrho$.*

Proof. If $\varrho(T) < \infty$, the Hahn-Banach theorem and theorem 3.1 imply that there is an S in $C_{\varrho'}$ such that $\varrho'(S) = 1$ and $\varrho(T) = \operatorname{tr} ST$. Therefore

$$\varrho(T) = \operatorname{tr} ST \leqslant |\operatorname{tr} ST| \leqslant \sup_{\varrho'(S)\leqslant 1} |\operatorname{tr} ST| = \varrho''(T).$$

Combining this with proposition 2.3, it follows that $\varrho(T) = \varrho''(T)$ for all $T \in C_\varrho$.

The abstract Lax-Milgram lemma can also be used to find linear functionals on reflexive Banach function spaces (cf. [1]).

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