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# ON THE TRANSITIVITY OF SOME RELATIONS BETWEEN ATLASES

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This paper is devoted to an examination of connections between some postulates concerning the class of mappings which give rise to the introduction of the following notions: of an atlas, of an equivalence of atlases, and of the maximal atlas among atlases equivalent to a given one (see [1]-[3]). To get correct definition of the equivalence of atlases we must introduce some restrictions concerning the class of mappings acting in the parameter space. Condition (3.1) of this paper may be considered as a new axiom of the theory of atlases. It enables us to introduce the concept of a structure being a generalization of the concept of a differentiable structure on a manifold. This generalization goes in two directions: it replaces the assumption that the space of parameters is a Euclidean space by the assumption that it is an arbitrary topological space, for example a Banach space. Moreover, it considers the family $\mathscr{C}$ instead of the family of all diffeomorphisms acting in the parameter space.

For a function $f$ and a set $B$, $D_f$ and $f^{-1}[B]$ will denote the domain of $f$ and counter-image of $B$ by $f$. If $A \subset D_f$, so we denote by $f|A$ and $f[A]$ the restriction of $f$ to $A$ and the image of $A$ by $f$, respectively. For given functions $g$ and $f$ we denote by $g\circ f$ the *composition* of $f$ and $g$, i.e. the function whose domain is the set $f^{-1}[D_g]$ and such that $(g\circ f)(x) = g\big(f(x)\big)$ for $x \in f^{-1}[D_g]$. For a given set $A$, $i_A$ denotes the identity function on $A$. For a topological space $X$ and for a set $A$ of its points we denote by $X|A$ the topological space induced in $A$ by $X$.

**1.** Any function $f$ such that $D_f$ is an open set of the topological space $X$, $f[D_f]$ is an open set of the topological space $P$ and $f$ is a homeomorphism of $X|D_f$ onto $P|f[D_f]$ will be called a *map* $X$ *with respect to* $P$.

Let there be given a space $P$ and a family $C$ of functions mapping the sets of points of $P$ into the sets of points of $P$, shortly: a family *acting in* $P$. A family $F$ of maps of the space $X$ with respect to $P$ is said to be an *atlas* of the class $\mathscr{C}$ of the space $X$ with respect to $P$ if and only if the following conditions are satisfied:

(1.1) $\bigcup_{f \in F} D_f$ is the set of all points of $X$

and

(1.2) for any $g, f \in F$ the function $g \circ f^{-1}$ belongs to $\mathscr{C}$.

We denote by $\mathfrak{A}(\mathscr{C}, X, P)$ the family of all such atlases.

**2.** We shall consider the relation $=_{\mathscr{C}XP}$ which is defined as follows: $F =_{\mathscr{C}XP} G$ if and only if $F, G$ and $F \cup G$ belong to $\mathfrak{A}(\mathscr{C}, X, P)$.

It is clear that the relation $=_{\mathscr{C}XP}$ is reflexive and symmetric in $\mathfrak{A}(\mathscr{C}, X, P)$, but it needs not to be transitive. The transitivity of this relation is very important for the uniqueness of prolongation of a given atlas to the maximal one.

**2.1.** *The relation $=_{\mathscr{C}XP}$ is transitive if and only if for every $F \in \mathfrak{A}(\mathscr{C}, X, P)$ there exists an $F_0$ such that*

(2.1) $F_0 \in \mathfrak{A}(\mathscr{C}, X, P)$ *and, for every $H$, if $F \subset H \in \mathfrak{A}(\mathscr{C}, X, P)$, then $H \subset F_0$.*

Proof. If the relation $=_{\mathscr{C}XP}$ is transitive, then for every $F \in \mathfrak{A}(\mathscr{C}, X, P)$ the set $F_0 = \bigcup \{G; G =_{\mathscr{C}XP} F\}$ satisfies condition (2.1).

Suppose now that for every $F \in \mathfrak{A}(\mathscr{C}, X, P)$ there exists an $F_0$ satisfying condition (2.1). Let $G =_{\mathscr{C}XP} F$, $F =_{\mathscr{C}XP} H$ and let $F_0$ satisfy (2.1). Then $G \cup F$ and $F \cup H$ belong to $\mathfrak{A}(\mathscr{C}, X, P)$. Thus $G \cup F \subset F_0$ and $F \cup H \subset F_0$. Therefore $G \subset G \cup H \subset F_0$. Hence it follows that $G \cup H \in \mathfrak{A}(\mathscr{C}, X, P)$.

**3.** In this section we consider a class $\mathscr{C}$ acting in $P$ and we study some conditions related to the following one:

(3.1) the relation $=_{\mathscr{C}XP}$ is transitive for every topological space $X$.

For every family $\mathscr{C}$ acting in $P$ we denote by $\mathscr{C}_P$ the set of all functions $\varphi \in \mathscr{C}$ such that $D_\varphi$ and $\varphi[D_\varphi]$ are open on $P$, $\varphi$ is a homeomorphism of $P|D_\varphi$ onto $P|\varphi[D_\varphi]$, $\varphi^{-1} \in \mathscr{C}$ and there exist sets $A$ and $B$ open in $P$ such that $D_\varphi \subset A$, $\varphi[D_\varphi] \subset B$, $i_A \in \mathscr{C}$ and $i_B \in \mathscr{C}$.

**3.1.** *For every family $\mathscr{C}$ acting in $P$ the family $\mathscr{C}_P$ is the smallest of families $\mathscr{C}'$ acting in $P$ and satisfying the condition*

(3.1.1) $\mathfrak{A}(\mathscr{C}, X, P) = \mathfrak{A}(\mathscr{C}', X, P)$ *for any topological space $X$.*

Proof. Let $\mathscr{C}$ act in $P$ and $X$ be a topological space. $\mathfrak{A}(\mathscr{C}_P, X, P) \subset \mathfrak{A}(\mathscr{C}, X, P)$, because $\mathscr{C}_P \subset \mathscr{C}$. Let $g, f \in F \in \mathfrak{A}(\mathscr{C}, X, P)$. Put $\varphi = g \circ f^{-1}$, $A = f[D_f]$, $B = g[D_g]$. Then $\varphi^{-1} = f \circ g^{-1} \in \mathscr{C}$, $f[D_f] \subset A$, $g[D_g \cap D_f] \subset B$, $i_A = f \circ f^{-1} \in \mathscr{C}$, $i_B = g \circ g^{-1} \in \mathscr{C}$, the sets $D_\varphi$, $\varphi[D_\varphi]$, $A$ and $B$ are open in $P$ and $\varphi$ is a homeomorphism of $P|D_\varphi$ onto $P|\varphi[D_\varphi]$. Thus $\varphi\in\mathscr{C}_P$. Therefore $F\in\mathfrak{A}(\mathscr{C}_P, X, P)$. So the set $\mathscr{C}' = \mathscr{C}_P$ fulfills condition (3.1.1).

Assume now that the family $\mathscr{C}'$ acting in $P$ fulfills (3.1.1) and consider an arbitrary function $\varphi\in\mathscr{C}_P$. Then there exist sets $A$ and $B$ open in $P$ such that $D_\varphi\subset A$, $\varphi[D_\varphi]\subset B$, $i_A\in\mathscr{C}$ and $i_B\in\mathscr{C}$. Let $E$ be a set disjoint with $A$ and such that $\operatorname{card}(E) = \operatorname{card}(B-\varphi[D_\varphi])$. Hence it follows that there exists a function $g$ defined on $D_\varphi\cap E$ such that $g|D_\varphi = \varphi$, $g|E$ is a one-to-one mapping and $g[E] = B-\varphi[D_\varphi]$.

Let us consider the topological space $X$, the set of all points of which is $A\cup E$ and the family of all its open sets is the family of all $G\subset A\cup E$ such that $g[G\cap(D_\varphi\cup E)]$ is open in $P|B$ and $G\cap A$ is open in $P|A$. It is easy to verify that $X|A = P|A$, $g$ is a homeomorphism of $X|(D_\varphi\cup E)$ onto $P|B$, $g\circ i_A^{-1} = g\circ i_A = \varphi$ and $\{g, i_A\}\in\mathfrak{A}(\mathscr{C}, X, P)$. Then $\{g, i_A\}\in\mathfrak{A}(\mathscr{C}', X, P)$. Moreover, $\varphi\in\mathscr{C}'$. Therefore we have got the inclusion $\mathscr{C}_P\subset\mathscr{C}'$.

It is clear that the equality in condition (3.1) is equivalent to the following statement: the relations $=_{\mathscr{C}XP}$ and $=_{\mathscr{C}'XP}$ are equal each other.

**3.2.** *If a family $\mathscr{C}$ acting in $P$ fulfills the conditions*

(3.2.1) *if $\psi, \varphi\in\mathscr{C}_P$, then $\psi\circ\varphi\in\mathscr{C}$,*

(3.2.2) *if $\mathscr{R}\subset\mathscr{C}_P$ and $\bigcup_{\varphi\in\mathscr{R}}\varphi$ is a one-to-one mapping, then $\bigcup_{\varphi\in\mathscr{R}}\varphi\in\mathscr{C}$,*

*then $\mathscr{C}$ satisfies condition* (3.1).

Proof. Let us assume that $\mathscr{C}$ satisfies (3.2.1) and (3.2.2). Let $X$ be an arbitrary topological space and let $G = {}_{\mathscr{C}XP}F = {}_{\mathscr{C}XP}H$. Consider arbitrary $g, h\in G\cup H$. If $g, h\in G$ or $g, h\in H$, so $h\circ g^{-1}\in\mathscr{C}$. Then it may be supposed that $g\in G$ and $h\in H$. From the fact that $F$ fulfills (1.1) it follows that

$$h\circ g^{-1} = \bigcup_{f\in F}(h\circ f^{-1})\circ(f\circ g^{-1}).$$

Applying (3.2.1) and (3.2.2) we obtain $h\circ g^{-1}\in\mathscr{C}$. Thus $F\cup H\in\mathfrak{A}(\mathscr{C}, X, P)$.

It is possible to give another simple system of postulates concerning the family $\mathscr{C}$ and assuring the fulfillment of (3.1).

**3.3.** *If a family $\mathscr{C}$ acting in $P$ satisfies the conditions*

(3.3.1) *if $\psi, \varphi\in\mathscr{C}_P$ and $D_\psi = \varphi[D_\varphi]$, then $\psi\circ\varphi\in\mathscr{C}$,*

(3.3.2) *if $\varphi\in\mathscr{C}_P$ and $A$ is a non-empty subset of $D_\varphi$, open in $P$, then $\varphi|A\in\mathscr{C}$,*

*then $\mathscr{C}$ satisfies* (3.2.1).

Proof. Let us suppose that $\mathscr{C}$ fulfills (3.3.1) and (3.3.2). Let $\psi, \varphi \in \mathscr{C}_P$. By (3.3.2) and the definition of the family $\mathscr{C}_P$ the functions $\psi_0 = \psi|(D_\psi \cap \varphi[D_\varphi])$ and $\varphi_0 = \varphi|\varphi^{-1}[D_\psi]$ belong to $\mathscr{C}_P$, $D_{\psi_0} = \varphi_0[D_{\varphi_0}]$ and $\psi\circ\varphi = \psi_0\circ\varphi_0$. Thus, by (3.3.1), we have $\psi\circ\varphi\in\mathscr{C}$.

It is clear that (3.2.1) implies (3.3.1). Consider the natural question: does any of the conditions (3.2.1), (3.2.2), (3.3.1) and (3.3.2) follow from (3.1)? We prove that none of them follows from (3.1). More precisely:

**3.4.** *If $P$ is a topological space and there exist its open sets $A$, $A'$, $B$ and $B'$ such that $A \neq A' \subset A$, $B \neq B' \subset B$, $A \cap B = \emptyset$ and the spaces $P|A'$ and $P|B'$ are homeomorphic, then there exists a family $\mathscr{C}$ acting in $P$ and satisfying* (3.1) *but none of the conditions* (3.2.1), (3.2.2), (3.3.1), *and* (3.3.2).

Proof. Let us suppose that $P$ satisfies the condition formulated in 3.4. Let $\varphi$ be a homeomorphism of $P|A'$ onto $P|B'$. Put

$$\mathscr{C} = \{\varphi, \varphi^{-1}, i_A, i_B\}. \tag{3.4.0}$$

From the fact that $i_{A'}\notin\mathscr{C}$ it follows that neither (3.3.1) nor (3.3.2) is satisfied. From $i_{A\cup B}\notin\mathscr{C}$ we obtain that $\mathscr{C}$ does not fulfill (3.2.2).

Let $X$ be an arbitrary topological space and $F\in\mathfrak{A}(\mathscr{C}, X, P)$. From (3.4.0) it follows that for every $g, f\in F$ one of the four cases

(3.4.1) $f[D_f] = A$ and $f = g$,

(3.4.2) $f[D_f] = B$ and $f = g$,

(3.4.3) $f[D_f] = A, g[D_g] = B$ and $g\circ f^{-1} = \varphi$,

(3.4.4) $f(D_f] = B, g[D_g] = A$ and $g\circ f^{-1} = \varphi^{-1}$

holds. From conditions (3.4.1)-(3.4.4) it immediately follows that the set $F$ cannot contain more than two elements.

Let $F =_{\mathscr{C}XP} G$. First, suppose that the set $F$ contains exactly one element, i.e., there exists a map $f$ such that $F = \{f\}$. It is easy to see that we may assume $f[D_f] = A$. Let us now suppose that $F \cup G \neq F$. Then, a map $g \neq f$ would exist such that $g\in G$. Hence $F \cup G = \{f, g\}$ and $g[D_g] = B$. So (3.4.3) is satisfied. Thus $g\circ f^{-1} = \varphi$. Then $g|(D_g \cap D_f) = \varphi\circ f$. From the fact that $F$ and $F \cup G$ are atlases it follows that $D_f = D_f \cup D_g$. In other words, $D_g \subset D_f$. Thus $g = \varphi\circ f$. So it would follow that $B = g[D_g] \subset \varphi[D_\varphi] = B'$. However this is impossible. Therefore $G \cup F = F$, and $G = F$.

Let us now suppose that $F$ contains exactly two elements. From the fact that $G =_{\mathscr{C}XP} F$ it follows that $G$ cannot contain exactly one element. From conditions (3.4.1)-(3.4.4), which are satisfied for any functions $g, f\in G \cap F$, it follows that $G = F$. So the relation $=_{\mathscr{C}XP}$ is transitive. Thus condition (3.1) is fulfilled.

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