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THE DECIDABILITY OF SOME $\aleph_0$-CATEGORICAL THEORIES

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Complete $\aleph_0$-categorical extensions of theories which are interpretable in the monadic second-order theory of two successor functions, hereinafter denoted by T_{s2s}, are investigated in this paper*. The theory T_{s2s} was shown by Rabin [2] to be decidable. This very powerful result yields the decidability of many other theories. Moreover, if a theory is interpretable in T_{s2s} (a notion to be made precise in Section 1), then the weak monadic second-order theory of its models and the monadic second order theory of its countable models are decidable. Let us call the logic that permits quantification over both finite and arbitrary subsets as weak/monadic second-order logic.

THEOREM 1 (Rabin). *If T is any theory which is interpretable in T_{s2s}, then the weak/monadic second-order theory of its countable models is decidable.*

The following theorem, which is proved in Section 2, is the main result of this paper.

THEOREM 2. *If T_0 is an $\aleph_0$-categorical completion of a theory which is interpretable in T_{s2s}, then T_0 is interpretable in T_{s2s}.*

As a consequence we see that any $\aleph_0$-categorical completion of a theory interpretable in T_{s2s} is decidable; indeed, even the weak/monadic second-order theory of its countable model is decidable. Examples of theories which are interpretable in T_{s2s} are: the theory of a single unary function, the theory of graphs with no circuits of length greater than n, and the theory of trees. These examples will be discussed in Section 3.

1. Preliminaries. A *tree* is a partially ordered set $(A, <)$ such that the set of predecessors of any element is linearly ordered by $<$. If $(A, <)$ is a tree and $U_0, \ldots, U_{n-1} \subseteq A$, then the structure $(A, <, U_0, \ldots, U_{n-1})$ will be called an *n-augmented tree.* There is a 2-augmented tree of special

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importance. Let

$$B = \bigcup\{\{0,1\}^n : n < \omega\},$$

and define the relation $<$ on B so that $a < b$ if a is a proper initial segment of b. For $i = 0, 1$, let

$$U_i = \{a^\frown\langle i\rangle : a \in B\}.$$

Then we call the structure $(B, <, U_0, U_1)$ the *full binary tree*. If $U_2, \ldots, U_{m+2} \subseteq B$, then the $(m+1)$-augmented tree

$$(B, <, U_0, \ldots, U_{m+2})$$

is called an *augmented full binary tree*.

We wish to use monadic second-order logic to discuss augmented trees. This logic allows quantification over subsets. There are two types of variables — individual variables denoted by lower case letters (e.g., $x_0, x_1, \ldots$), and set variables denoted by upper case letters (e.g., $X_0, X_1, \ldots$). For this logic, two-sorted structures $(\mathfrak{A}, \mathfrak{S})$ are appropriate, where $\mathfrak{A}$ is an augmented tree, $\mathfrak{S} \subseteq \mathfrak{P}(A)$, and each subset of A definable in $(\mathfrak{A}, \mathfrak{S})$ by a monadic second-order formula is in $\mathfrak{S}$. We call such an $(\mathfrak{A}, \mathfrak{S})$ a *second-order augmented tree*. For example, if $\mathfrak{A}$ is an augmented tree, then $(\mathfrak{A}, \mathfrak{P}(A))$ is a second-order augmented tree. If $\mathfrak{B}$ is the full binary tree, then $(\mathfrak{B}, \mathfrak{P}(B))$ is the second-order full binary tree, and its theory is denoted by T_{s2s}, which is (essentially) the monadic second-order theory of two successor functions (see [2]). Any second-order augmented tree $(A, <, U_0, \ldots, U_{m-1}, \mathfrak{S})$, for which $(A, <, U_0, U_1, \mathfrak{S})$ is a model of T_{s2s}, is a second-order m-augmented binary tree.

We now make precise what we mean by a theory being interpretable in T_{s2s}. Let φ be a (monadic second-order) formula. We say that φ is *n-ary* if the only free variables are individual variables, and they are among $x_0, \ldots, x_{n-1}$. Now let $\varrho = \{R_1, \ldots, R_r\}$ be a finite similarity type, where each R_i is an n_i-ary relation symbol. Let $\pi = \langle \pi_0, \ldots, \pi_r\rangle$ be such that π_0 is a 1-ary formula and every other π_i is an n_i-ary formula in the language of second-order m-augmented trees.

Now suppose that $(\mathfrak{A}, \mathfrak{S})$ is a second-order m-augmented tree in which $(\mathfrak{A}, \mathfrak{S}) \models \exists x_0 \pi_0$. Then $(\mathfrak{A}, \mathfrak{S})^\pi$ is the ϱ-structure $\mathfrak{B}$, where

$$B = \{a \in A : (\mathfrak{A}, \mathfrak{S}) \models \pi_0(a)\},$$

and for $1 \leqslant i \leqslant r$

$$R_i^{\mathfrak{B}} = \{\bar{a} \in B^{n_i} : (\mathfrak{A}, \mathfrak{S}) \models \pi_i(\bar{a})\}.$$

Then we say that a (first-order) ϱ-theory T is *interpretable in T_{s2s}* by π if $\mathfrak{B}$ is a countable model of T iff $\mathfrak{B} \cong (\mathfrak{A}, \mathfrak{S})^\pi$ for some countable second-order m-augmented tree $(\mathfrak{A}, \mathfrak{S})$.

2. Proof of Theorem 2. The proof uses a refinement of the method of [4], wherein it is proved that each $\aleph_0$-categorical theory of trees is decidable.

We begin by defining, for each $p < \omega$, the class Q_p which consists of formulas in the language of second-order augmented trees. Let us first slightly augment this language so as to include terms of the following type: $X \cup Y$, $X \cap Y$, X', $\{y: y \geqslant x\}$, and $\{y: y \leqslant x\}$. These terms are to be given their natural interpretations. Then let Q_p be the class of formulas in which the number of distinct bound (individual and set) variables is not greater than p. Notice that for each $m < \omega$ the class of all m-ary formulas in Q_p is finite modulo logical equivalence.

For a second-order augmented tree $(\mathfrak{A}, \mathfrak{S})$ and elements $a_0, \ldots, a_{m-1} \in A$, define $\varphi^p_{a_0,\ldots,a_{m-1}}$ to be the conjunction of those m-ary formulas φ in Q_p such that

$$(\mathfrak{A}, \mathfrak{S}) \models \varphi(a_0, \ldots, a_{m-1}).$$

Now let $(\mathfrak{A}, \mathfrak{S})$ be a second-order augmented binary tree. Let $a \in A$, and let $X \subseteq A$ be non-empty and finite. Then there is a subset $Y \subseteq X$ with at most 3 elements satisfying:

(1) there is $y \in Y$ such that whenever $x \in X$ and $z \leqslant x, a$, then $z \leqslant y, a$;

(2) there are $y_1, y_2 \in Y$ such that whenever $x_1, x_2 \in X$, $z \leqslant y_1, y_2$ and $a \leqslant x_1, x_2$, then $z \leqslant x_1, x_2$.

Such a Y will be called a *nucleus of X for a*. The importance of a nucleus is revealed in the following lemma:

LEMMA. *Suppose that $(\mathfrak{A}, \mathfrak{S})$ is a second-order augmented binary tree, $X = \{a_0, \ldots, a_n\} \subseteq A$, $a \in A$, and $Y = \{a_i, a_j, a_k\}$ is a nucleus of X for a. If $p < \omega$ and $b_0, \ldots, b_n$, $b \in A$ are such that*

$$\varphi^p_{a_0,\ldots,a_n} = \varphi^p_{b_0,\ldots,b_n} \quad \textit{and} \quad \varphi^p_{a_i,a_j,a_k,a} = \varphi^p_{b_i,b_j,b_k,b},$$

then

$$\varphi^p_{a_0,\ldots,a_n,a} = \varphi^p_{b_0,\ldots,b_n,b}.$$

The proof can be easily completed by a back-and-forth argument.

We give an immediate consequence of the Lemma. Let $p < \omega$ and let $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in A$ be such that if $i, j, k, m < n$, then

$$\varphi^p_{a_i,a_j,a_k,a_m} = \varphi^p_{b_i,b_j,b_k,b_m}.$$

Then $\varphi^p_{a_0,\ldots,a_{n-1}} = \varphi^p_{b_0,\ldots,b_{n-1}}$. (It is easily seen that it suffices to assume only that $\varphi^p_{a_i,a_j,a_k} = \varphi^p_{b_i,b_j,b_k}$.)

We now proceed with the proof of Theorem 2. Let T be interpretable in T_{s2s} by $\pi = \langle \pi_0, \ldots, \pi_r \rangle$, where each π_i is an n_i-ary formula. Let $(\mathfrak{A}, \mathfrak{S})$ be a countable second-order augmented binary tree such that $\mathfrak{B} = (\mathfrak{A}, \mathfrak{S})^\pi$

is a model of T_0. Choose $p < \omega$ large enough so that each $\pi_i \in Q_p$, and if $n \geqslant 3$ and if $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in B$ are such that $\langle a_0, \ldots, a_{n-1}\rangle$ and $\langle b_0, \ldots, b_{n-1}\rangle$ realize different types in $\mathfrak{B}$, then there are $i < j < k < n$ such that $\varphi^p_{a_i,a_j,a_k} \neq \varphi^p_{b_i,b_j,b_k}$. The existence of such a p depends on the mentioned consequence of the Lemma, and on Ryll-Nardzewski's well--known characterization (in [3]) of $\aleph_0$-categorical theories. Now let σ be the conjunction of the universal closures of the following formulas:

(1) $\qquad \varphi^p_{a_0,a_1,a_2} \to \exists x_3 \varphi^p_{a_0,a_1,a_2,a_3}, \quad$ where $a_0, a_1, a_2, a_3 \in B$.

(2) $\quad \varphi^p_{a_0,a_1,a_2} \to \forall x_3 \big[\pi_0(x_3) \to \bigvee\{\varphi^p_{a_0,a_1,a_2,a_3} \colon a_3 \in B\}\big], \quad$ where $a_0, a_1, a_2 \in B$.

(3) $\qquad \exists x_0 \varphi^p_a, \quad$ where $a \in B$.

Obviously, $(\mathfrak{A}, \mathfrak{S}) \models \sigma$. Let $\pi' = \langle \sigma \wedge \pi_0, \pi_1, \ldots, \pi_r\rangle$. To conclude the proof it suffices to show that if $(\mathfrak{A}', \mathfrak{S}')$ is a countable second-order augmented binary tree such that $(\mathfrak{A}', \mathfrak{S}') \models \exists x_0(\sigma \wedge \pi_0)$, then $\mathfrak{B} \cong \mathfrak{B}'$, where $\mathfrak{B}' = (\mathfrak{A}', \mathfrak{S}')^{\pi'}$. This is easily accomplished by a back-and-forth argument. To this end, suppose that $a_0, \ldots, a_{n-1} \in B$ and $b_0, \ldots, b_{n-1} \in B'$ are such that $\varphi^p_{a_0,\ldots,a_{n-1}} = \varphi^p_{b_0,\ldots,b_{n-1}}$. Then

(i) if $a_n \in B$, then there is $b_n \in B'$ such that $\varphi^p_{a_0,\ldots,a_n} = \varphi^p_{b_0,\ldots,b_n}$;

(ii) if $n > 0$ and $b_n \in B'$, then there is $a_n \in B$ such that $\varphi^p_{a_0,\ldots,a_n} = \varphi^p_{b_0,\ldots,b_n}$.

In case (i), for $n = 0$ choose $b_0 \in B'$ such that $(\mathfrak{A}', \mathfrak{S}') \models \varphi^p_{a_0}(b_0)$ (as allowed by (3)). For $n > 0$ let $\{a_i, a_j, a_k\}$ be a nucleus of $\{a_0, \ldots, a_{n-1}\}$ for a. Using sentences (1), let $b_n \in B'$ be such that

$$(\mathfrak{A}', \mathfrak{S}') \models \varphi^p_{a_i,a_j,a_k,a_n}(b_i, b_j, b_k, b_n).$$

In case (ii), let $\{b_i, b_j, b_k\}$ be a nucleus of $\{b_0, \ldots, b_{n-1}\}$ for b_n. Then, by sentences (2), there exists $a_n \in B$ such that

$$\varphi^p_{a_i,a_j,a_k,a_n} = \varphi^p_{b_i,b_j,b_k,b_n}.$$

By the Lemma, this a_n works.

3. Applications. We give examples of several theories which are interpretable in T_{s2s}.

The theory of a single unary function is interpretable in T_{s2s} (see Section 2.2 of [2]). Since we technically do not allow function symbols, we could consider the more general theory with the one binary relation symbol R which is axiomatized by the sentence

$$\forall xyz\big(R(x, y) \wedge R(x, z) \to y = z\big).$$

The $\aleph_0$-categorical completions of the theory of a single unary function were studied in [5].

The $\aleph_0$-categorical completions of the theory of graphs with no circuits of length greater than n were characterized in [1]. It is not hard to see from the analysis given there that this theory is interpretable in T_{s2s}.

Finally, we consider the theory of trees, all $\aleph_0$-categorical completious of which were shown in [4] to be decidable. We show that the theory of trees is interpretable in T_{s2s}. Let $\mathfrak{B}$ be the full binary tree, and let

$$A = \{0\}\cup\{s^\frown\langle 0, 0, 1, 1\rangle : s \in B\}.$$

Then $\mathfrak{B}|A$ is a tree in which each element has an infinity of immediate successors. The relation $s \prec t$ holds on A if either $s < t$ or else

(1) whenever n is the largest natural number such that $s|n = t|n$, then $s|n^\frown\langle 1\rangle \leqslant t$; and

(2) whenever $x \in A$ and $x < s$, then $x < t$.

Then $(A, \prec)$ is easily seen to be a universal tree in the sense that each countable tree is embeddable in it, and the interpretability follows immediately.

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