COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 1

GENERALIZED ALMOST-CONVERGENCE VS. MATRIX SUMMABILITY

BY

ROBERT E. ATALLA (ATHENS, OHIO)

1. Introduction. Let $T = (t_{mn})$ be a non-negative regular matrix, considered as a linear operator on $C_b(N)$, the set of bounded real functions on the positive integers N. Let C_T be the bounded convergence field of T, i.e., the set of $f \in C_b(N)$ such that $\lim\limits_m \sum t_{mn}f(n)$ exists. A *generalized almost-convergence field* with respect to T is defined as follows. A T-invariant mean is a positive linear functional m on $C_b(N)$ such that

(i) $m(1) = 1$,

(ii) $m(Tf) = m(f)$ $(f \in C_b(N))$,

(iii) if F is a finite subset of N and 1_F its indicator function, then $m(1_F) = 0$.

((iii) ensures that the Borel measure on βN corresponding to m will be supported by $\beta N \setminus N$.)

We say that f is *almost-convergent* with respect to T and write $f \in \mathrm{ac}_T$ if $m_1(f) = m_2(f)$ for any two T-invariant means m_1 and m_2. If R is the shift matrix $Rf(n) = f(n+1)$, then ac_R is the usual space of almost-convergent sequences.

In [10], p. 187, it was shown that ac_R is not equal to the convergence field C_S of any regular matrix S. It is natural to ask under what conditions the equation $\mathrm{ac}_T = C_S$ is possible. Is it ever possible? Yes. For instance, if $T^2 = T$, then it is not hard to show that $\mathrm{ac}_T = C_T$. More generally, if T is *strongly ergodic*, i.e., there exists a matrix projection Q such that

$$T_n f \equiv (1/n)(I + \ldots + T^{n-1})f \to Qf \quad \text{for all } f \in C_b(N),$$

then $\mathrm{ac}_T = C_Q$. Thus some sort of ergodicity of T is sufficient for $\mathrm{ac}_T = C_S$ to hold. In this paper we show that if the matrix S is required to satisfy certain hypotheses, then ergodicity is necessary as well. Assume S and T are Markov operators on $C(\beta N \setminus N)$ induced by non-negative regular matrices (see Section 4). Assume S is "good" as defined in 2.6 below. This paper is devoted to proving the following

MAIN THEOREM. *If* $\mathrm{ac}_T = C_S$, *then there exists a matrix-induced Markov projection* Q *on* $C(\beta N\setminus N)$ *such that* $\|T_n - Q\| \to 0$ (*i.e.* T *is uniformly ergodic*).

1.1. Outline. In Section 2, we develop a relation between S and T which will enable us to show later that T is "locally strongly ergodic". In a preliminary effort [5], the author was able to show that if S is assumed to be a projection, then the relation between S and T holds, and local ergodicity of T can be proved (see the remarks at the beginning of Section 2). "Good operators", as defined in 2.6, are sufficiently projection-like, so that we can again prove local ergodicity of T by appealing to a more general result of Sine [13]. In Section 3 we show that, for our particular situation of operators on $C(\beta N\setminus N)$, local strong ergodicity implies not only global strong ergodicity, but even uniform ergodicity. In Section 4 we prove the Main Theorem.

1.2. Notation. In Sections 2 and 3, X is an arbitrary compact space, and S, T are Markov operators on $C(X)$, i.e., positive linear operators with $T1 = S1 = 1$. By $C(X)'$ we denote the dual space of $C(X)$, i.e., the space of regular Borel measures on X, T' is the adjoint of T, and $\boldsymbol{P}$ is the weak-* compact convex set of regular Borel probabilities on X. Let $\boldsymbol{P}_T = \{m \in \boldsymbol{P}: T'm = m\}$. We note that, with the exception of Lemma 3.6, all the results of Sections 2 and 3 are valid for this general situation. In Section 4 we specialize to matrix-induced Markov operators with state space $X = \beta N\setminus N$.

2. The condition $S'\boldsymbol{P} = \boldsymbol{P}_T$. As we shall see in Section 4, the condition $\mathrm{ac}_T = C_S$ for matrices translates into the condition $T_n f \to 0$ iff $Sf = 0$. Now this last condition implies easily $S'\boldsymbol{P} \subset \boldsymbol{P}_T$. (First, it implies $ST = S$, because $T_n(Tf - f) \to 0$ for all f, whence $S(Tf - f) = 0$. Secondly, $ST = S$ iff $S'\boldsymbol{P} \subset \boldsymbol{P}_T$, because $ST = S$ iff $T'S' = S'$ iff $T'(S'm) = S'm$ for all $m \in \boldsymbol{P}$.) In [5], Lemma 2.2, it is shown that if $S^2 = S$, then $S'\boldsymbol{P} = \boldsymbol{P}_T$, and this relation is basic in deriving conclusions about the local ergodic behavior of T. In this section we seek to define a larger class of operators S for which this holds. More specifically, we wish to reverse the implications of 2.2 below.

2.1. Definition. $\boldsymbol{F}_T = \{m \in C(X)': T'm = m\}$.

Consider the following three conditions:

(i) $S'\boldsymbol{P} = \boldsymbol{P}_T$,

(ii) $T_n f \to 0$ iff $Sf = 0$,

(iii) $\boldsymbol{F}_T$ is the weak-* closure of $S'(C(X)')$.

2.2. PROPOSITION. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii). Suppose $T_n f \to 0$. Since, as noted above, (i) implies $ST = S$, we get $Sf = ST_n f \to 0$, i.e., $Sf = 0$. Conversely, if $Sf = 0$, then

$0 = S'm(f)$ for all $m \in \boldsymbol{P}$, whence $m_0(f) = 0$ for all $m_0 \in \boldsymbol{P}_T$. Now it is well known that $\boldsymbol{P}_T$ is the set of cluster points of functionals of the form $T'_n m$, where $m \in \boldsymbol{P}$ and T'_n is the adjoint on T_n. Hence it follows easily that $m_0(f) = 0$ for all $m_0 \in \boldsymbol{P}_T$ iff $T_n f \to 0$ uniformly.

(ii) $\Rightarrow$ (iii). Condition (ii) implies $S'\boldsymbol{P} \subset \boldsymbol{P}_T$, and hence $S'(C(X)') \subset \boldsymbol{F}_T$. Suppose $m \in \boldsymbol{F}_T$ is not in the closure of $S'(C(X)')$. Then $m(f) = 1$ for some $f \in C(X)$ and $0 = S'm_0(f) = m_0(Sf)$ for all $m_0 \in C(X)'$. Hence $Sf = 0$ and, by (ii), $T_n f \to 0$, whence $0 = \lim m(T_n f) = m(f) = 1$, a contradiction.

2.3. LEMMA. *Let S be a Markov operator on $C(X)$ such that $S(C(X)^+)$ is dense in $S(C(X))^+$. Then $S'm \geqslant 0$ implies that there exists $m_0 \geqslant 0$ such that $S'm = S'm_0$.*

Proof. Let $0 \neq S'm \geqslant 0$. If $f \in S(C(X))^+$, there exist $g_n \in C(X)^+$ such that $Sg_n \to f$, whence

$$m(f) = \lim_n m(Sg_n) = \lim_n S'm(g_n) \geqslant 0.$$

Thus m restricted to $\operatorname{cl} S(C(X))$ is a positive linear functional. Let $n = m(1)^{-1}m$ and let n_0 be a Hahn-Banach extension to $C(X)$ of n restricted to $\operatorname{cl} S(C(X))$ with norm 1. Since $n_0(1) = 1$, n_0 is a probability measure, and we must show that $S'n_0 = S'n$ (whence $S'm = S'(m(1)n_0)$). But $n - n_0$ vanishes on $S(C(X))$, so if $f \in C(X)$, then

$$0 = (n - n_0)(Sf) = S'n(f) - S'n_0(f).$$

2.4. COROLLARY. *If S is a closed range Markov operator such that $S(C(X)^+) = S(C(X))^+$, then the following are equivalent:*

(i) $S'\boldsymbol{P} = \boldsymbol{P}_T$,
(ii) $T_n f \to 0$ *iff* $Sf = 0$,
(iii) $\boldsymbol{F}_T = S'(C(X)')$.

Proof. Note that since S has closed range, S' has weak-* closed range ([11], Theorem 4.14). By 2.2 we need only to prove that (iii) $\Rightarrow$ (i). Clearly, (iii) implies $S'\boldsymbol{P} \subset \boldsymbol{P}_T$. Conversely, if $m \in \boldsymbol{P}_T$, then $m = S'n$ for some $n \in C(X)'$. Since $m \geqslant 0$, we infer from Lemma 2.3 that there exists $m_0 \geqslant 0$ such that $m = S'm_0$. To show $m_0 \in \boldsymbol{P}$, observe that $m_0(1) = m_0(S1) = S'm_0(1) = m(1) = 1$.

2.5. Remarks. The condition in 2.4 is restrictive. Clearly, every projection (Markov or not) satisfies it. On the other hand, suppose S is a one-one Markov operator having this property. Then the inverse is a Markov operator on $S(C(X))$, and hence S is an isometry. The following definition gives a broad class of suitable operators. (A detailed discussion of this condition and operators which satisfy it is given in [6].)

2.6. Definition. Let S be a Markov operator, let s_x $(x \in X)$ be the Borel probability representing $S'\delta_x$, and put

$$K = K \quad = \mathrm{cl}\bigcup\{\mathrm{supp}(s_x): x \in X\}.$$

Let A be the set of $f \in C(X)$ which is constant on $\mathrm{supp}(s_x)$ whenever s_x is an extreme point of the compact convex set $S'P$.

S is called *good* if it has closed range and there exists a Markov projection Q on $C(K)$ whose kernel is $\ker(S)_K = \{f|K: f \in \ker(S)\}$ and whose range is $A_K = \{f|K: f \in A\}$.

2.7. Examples. 1. A Markov projection is good, for if $S^2 = S$ and s_x is an extreme point in $S'P$, then Sf is constant on $\mathrm{supp}(s_x)$ for each $f \in C(X)$ ([14], Theorem 1.11). Now define Q on $C(K)$ as follows: for $f \in C(K)$ let f_0 be a continuous extension to X and let $Qf = (Sf)_0|K$.

2. A generalized averaging operator [3] is good if its range is closed. If we modify the definition of that paper so that the projection therein called S has range in A instead of M_T, we get a broader class of good operators.

Further examples are given in [6].

2.8. PROPOSITION. *Let S be a good Markov operator on $C(X)$. Then*

(i) *If $f \in C(K)$, then $S((Qf)_0) = S(f_0)$, where f_0 and $(Qf)_0$ are continuous extensions of f and Qf, respectively.*

(ii) $S(C(X)^+) = S(C(X))^+$.

Proof. (i) $Qf - f \in \ker(Q) = \ker(S)_K$, so it follows easily that $(Qf)_0 - f_0 \in \ker(S)$, whence the result.

(ii) Clearly, $S(C(X)^+) \subset S(C(X))^+$. Conversely, if $Sf \geqslant 0$, let $f_1 = f|K$. Then $Sf = S((f_1)_0) = S((Qf_1)_0)$. If $s_x = S'\delta_x$ is an extreme point of $S'P$, then Qf_1 is some constant value k on $\mathrm{supp}(s_x)$. Then $Sf(x) = S((Qf_1)_0)(x) = \int Qf_1 ds_x = k$, so that $Qf_1(y) = Sf(x) \geqslant 0$ for all y in $\mathrm{supp}(s_x)$. Now all extreme points of $S'P$ are of the form s_x for some $x \in X$ ([5], Lemma 2.3), and the Krein-Milman theorem implies that the union of these supports is dense in K. Hence $Qf_1 \geqslant 0$ on K. Let g be any non-negative continuous extension of Qf_1 to X. Then $Sf = Sg$ with $g \geqslant 0$, so $Sf \in S(C(X)^+)$.

It is a happy coincidence that good operators have another property crucial to the Main Theorem.

2.9. PROPOSITION. *Let S be a good Markov operator and let P_0 be the weak-* compact convex set of regular Borel probabilities supported by K. Then each extreme point of $S'P$ is an extreme point of $Q'P_0$, where Q is as in* 2.6.

Proof. As noted before, extreme points of $S'P$ take the form $s_x = S'\delta_x$, where $x \in X$. Now s_x is an element of $Q'P_0$; namely, if $y \in \mathrm{supp}(s_x)$, then $Q'\delta_y = s_x$, since $Qf(y) = s_x(f)$ for all $f \in C(K)$, as shown in the proof of

Proposition 2.8. If s_x is not an extreme point in $Q'\boldsymbol{P}_0$, then $s_x = tQ'm_1 + (1-t)Q'm_2$, where $0 < t < 1$, m_1 and m_2 are in $\boldsymbol{P}_0$, and s_x is equal neither to $Q'm_1$ nor to $Q'm_2$. Since $s_x = Q'\delta_y$ whenever $y \in \mathrm{supp}(s_x)$, and $\mathrm{supp}(Q'm_1) \subset \mathrm{supp}(s_x)$, we have for all f in $C(K)$

$$Q'm_1(f) = (Q')^2 m_1(f) = (Q'm_1)(Qf)$$

$$= \int Qf(y)dQ'm_1(y) = \int s_x(f)dQ'm_1(y) = s_x(f).$$

Hence $Q'm_1 = s_x$, a contradiction.

3. Some operator theory. The condition $S'\boldsymbol{P} = \boldsymbol{P}_T$ and Proposition 2.9 will enable us to prove later that the restriction of T to the set

$$M = M_T = \mathrm{cl} \bigcup \{\mathrm{supp}(m) : m \in \boldsymbol{P}_T\}$$

is strongly ergodic. We then use properties peculiar to Markov operators induced by regular matrices to infer not only global strong ergodicity, but even uniform ergodicity. In this section we develop these peculiar properties.

3.1. Definition. A closed set K is called a *P-set* if the countable intersection of neighborhoods of K is again a neighborhood of K. (P-sets enter summability theory via the Henriksen-Isbell theorem: the "support set" in $\beta N \setminus N$ of a regular matrix is a P-set; see [8] or [4].)

3.2. LEMMA. *Let K be a P-set and $\{m_k\}$ a countable set of positive regular Borel measures. Assume X is totally disconnected. Then there exists a clopen set $B \supset K$ such that $m_k(B) = m_k(K)$ for all k.*

Proof. Fix k. By regularity there exist open sets $V_n \supset K$ such that $m_k(V_n) < m_k(K) + 1/n$. Since K is a P-set, there exists an open set V^k such that $K \subset V^k \subset V_n$ for all n, and clearly $m_k(V^k) = m_k(K)$. Again, there exists an open set V such that $K \subset V \subset V^k$ for all k. Since X is totally disconnected and K compact, we may assume V is clopen.

3.3. LEMMA. *Let T be a Markov operator such that $M = M_T$ is a P-set. Assume X is totally disconnected. Then $(T^n)'\delta_x(X \setminus M) \to 0$ uniformly in $x \in X$.*

Proof. Write $A = X \setminus M$ and $(T^n)'\delta_x(A) = T^n 1_A(x)$. Now for x in M we have $\mathrm{supp}(T'\delta_x) \subset M$ ([14], Theorem 1.3), so $T'\delta_x(M) = 1$. Thus $1_M \leqslant T1_M$ or $T1_A \leqslant 1_A$. For each x, $T^n 1_A(x)$ is decreasing with n. To show the limit is 0, let m be a weak-* cluster point of $\{T_n'\delta_x : n = 1, 2, \ldots\}$, so that $m \in \boldsymbol{P}_T$. Since A is a "co-P-set", Lemma 3.2 implies that there exists a clopen set $B \subset A$ such that $T^n 1_B(x) = T^n 1_A(x)$ for all n. Let $\{T'_{n(i)}\delta_x : i \in I\}$ be a subnet with $T'_{n(i)}\delta_x \to m$ (weak-*). Since 1_B is in $C(X)$, we have

$$\lim_n T^n 1_A(x) = \lim_n T^n 1_B(x) = m(B) = 0$$

because $\operatorname{supp}(m) \subset M$ and $B \subset X \setminus M$.

Suppose the convergence is not uniform. Then there exist $\varepsilon > 0$ and x_n such that $T^n 1_A(x_n) \geqslant \varepsilon$ for all n. Let x be a cluster point of $\{x_n\}$, and $\{x_{n(i)}\}$ a subnet with $x_{n(i)} \to x$. There exists a clopen set $B \subset A$ with $T^n 1_B(x_k) = T^n 1_A(x_k)$ for all n and k, and also $T^n 1_B(x) = T^n 1_A(x)$ for all n. Since $\delta_{x(n(i))} \to \delta_x$ (weak-*) and 1_B is in $C(X)$, we have, for each fixed n,

$$T^n 1_A(x) = T^n 1_B(x) = \lim_i T^n 1_B(x_{n(i)}) = \lim_i T^n 1_A(x_{n(i)})$$
$$\geqslant \overline{\lim_i}\, T^{n(i)} 1_A(x_{n(i)}) \geqslant \varepsilon.$$

But this contradicts the fact that $T^n 1_A(x) \to 0$. (This sort of result is of some interest in ergodic theory — see, e.g., Theorem 1 of [9].)

3.4. Remark. Since M is an invariant set, T induces a Markov operator on $C(M)$. We say T is *locally strongly ergodic* if this induced operator is strongly ergodic [13].

3.5. Proposition. *Assume X is totally disconnected. If T is a locally strongly ergodic operator such that M is a P-set, then T is (globally) strongly ergodic.*

Proof. If $f \neq 0$ is in $C(X)$, we must show that $\{T_n f\}$ is Cauchy in the uniform norm. Let $\varepsilon > 0$. By Lemma 3.3 we can choose k so that $(T^k)'\delta_x(A) < \varepsilon/(8\|f\|)$ for all x in X, where $A = X \setminus M$. Since $T_n(T^k - I)f$ tends uniformly to 0 and $T_n f|M$ is Cauchy in the uniform norm on $C(M)$, there exists N such that the inequality $m, n \geqslant N$ implies

(1) $\|T_n T^k f - T_n f\| < \varepsilon/4$,

(2) for all y in M, $|T_n f(y) - T_m f(y)| < \varepsilon/4$.

We have

$$\|T_n f - T_m f\| \leqslant \|T_n f - T_n T^k f\| + \|T_n T^k f - T_m T^k f\| + \|T_m T^k f - T_m f\|$$
$$< \varepsilon/2 + \|T^k(T_n - T_m)f\|.$$

To show $\|T^k(T_n - T_m)f\| \leqslant \varepsilon/2$, we observe that for any x in X

$$|T^k(T_n f - T_m f)(x)| \leqslant \int_M |T_n f(y) - T_m f(y)|\, dt_x^k(y) + \int_A |T_n f - T_m f|\, dt_x^k$$
$$\leqslant (\varepsilon/4) t_x^k(M) + 2\|f\| t_x^k(A) \leqslant \varepsilon/4 + 2\|f\|\varepsilon/(8\|f\|) = \varepsilon/2,$$

where $t_x^k = (T^k)'\delta_x$.

3.6. Lemma. (i) *If R_k and R are matrix-induced operators on $C(\beta N \setminus N)$, and $R_k \to R$ in the strong operator topology, then $\|R_k - R\| \to 0$.*

(ii) *The space of matrix-induced operators on $C(\beta N \setminus N)$ is sequentially complete in the strong operator topology.*

Proof. For the proof of (i) see [12], p. 322, or [1], Lemma 2.3.

(ii) Let $\{R_k\}$ be Cauchy in the strong operator topology. We show that this implies the sequence is Cauchy in the uniform topology. If $\{R_k\}$ is not uniformly Cauchy, then there exist $\varepsilon > 0$ and $n(1) < n(2) < \ldots$ such that $\|R_{n(k)} - R_{n(k+1)}\| > \varepsilon$. By (i), there exists f in $C(\beta N \setminus N)$ such that $\|R_{n(k)}f - R_{n(k+1)}f\| \not\to 0$. But this contradicts the assumption that $\{R_k\}$ is Cauchy in the strong operator topology. Since $\{R_k\}$ is uniformly Cauchy, it converges uniformly to some R which, as can be easily seen, is induced by some regular matrix.

4. Proof of the Main Theorem. First we must show how matrices induce operators on $C(\beta N \setminus N)$. Let f' be the extension of $f\,(f \in C_b(N))$ to βN, and let f^* be the restriction of f' to the compact set $\beta N \setminus N$. If $V \subset N$, then $V' = \mathrm{cl}\, V$ in βN, and $V^* = V' \cap \beta N \setminus N$. The set V^* is not empty iff V is infinite, and sets of the form V^* $(V \subset N)$ are a basis of clopen sets for the topology of $N^* = \beta N \setminus N$.

$C(N^*)$, the space of continuous real functions of N^*, is isometric to the quotient space $C_b(N)/c_0$, where c_0 is the space of real functions on N with limit 0. If $T = (t_{mn})$ is a regular matrix, then $T(c_0) \subset c_0$, and hence T induces an operator T^* on $C(N^*)$ by the formula $T^*f^* = (Tf)^*$ $(f \in C_b(N))$. If T is non-negative, then T^* is a Markov operator, i.e., $T^* \geqslant 0$ and $T^*1 = 1$ (see [2]-[4]).

If $(C_T)^* = \{f^*: f \in C_T\}$, then $(C_T)^* = \{f \in C(N^*):\ Tf = \text{const}\}$. Likewise $(\mathrm{ac}_T)^* = \{f^*: f \in \mathrm{ac}_T\}$. We regard the T-invariant means as a certain compact convex set of regular Borel probabilities on N^* which are invariant under the adjoint of T^*. Letting $X = N^*$, we see that in Notation 1.2 this is just $\boldsymbol{P}_{T^*}$. If f is in ac_T and the value assigned by each invariant mean to f is k, then $\int f^* dm = k$ for all m in $\boldsymbol{P}_{T^*}$ (see [1]).

Since from now on we deal exclusively with the induced operator T^*, we drop the notational distinction between T and T^*, and just write T. Likewise we write $C_T = \{f \in C(N^*):\ Tf = \text{const}\}$ and $f \in \mathrm{ac}_T$ iff $\int f dm_1 = \int f dm_2$ for all m_1 and m_2 in $\boldsymbol{P}_T$. The following lemma enables us to apply the results of Section 2 to matrix-induced operators.

4.1. LEMMA. *If S and T are Markov operators on $C(N^*)$ induced by non-negative regular matrices, then $\mathrm{ac}_T = C_S$ iff $\{f\colon\ T_n f \to 0$ uniformly$\}$ $= \ker(S)$.*

Proof. The necessity is an easy exercise. To prove the sufficiency, assume $\mathrm{ac}_T = C_S$. Suppose, for some f, $T_n f \to 0$, but $Sf = k \neq 0$. By [2], Lemma 2.4, there exists g in $C(N^*)$ such that, for all h in $C(N^*)$, $T_n(gh) = T_n(g)T_n(h)$ for all n, $S(gh) = SgSh$, and Sg is non-constant. Then $T_n(gf) \to 0$ while $S(gf) = kSg$ is non-constant. Thus $fg \in \mathrm{ac}_T \setminus C_S$, a contradiction. Now suppose that $Sf = 0$ for some f, but $T_n f \to k \neq 0$. Let $h = k - f$ and apply the previous case.

4.2. Proof of the Main Theorem. We assume $\mathrm{ac}_T = C_S$, where S is good. By 4.1, 2.4, and 2.8, $S'\boldsymbol{P} = \boldsymbol{P}_T$. By 2.9 and the Krein-Milman theorem, there exists a Markov projection Q on $C(M)$ such that $\boldsymbol{P}_T = S'\boldsymbol{P} \subset Q'\boldsymbol{P}_0$ and each extreme point of $\boldsymbol{P}_T$ is an extreme point of $Q'\boldsymbol{P}_0$. Now Q is *continuously scattered*, i.e., there exists a family of continuous functions, each constant on the support of each extreme Q-invariant probability, which separates the extreme invariant probabilities ([13], Theorem 3). The operator induced by T on $C(M)$ is also continuously scattered, and, again by [13], Theorem 3, T is locally strongly ergodic. Since $S'\boldsymbol{P} = \boldsymbol{P}_T$, the set

$$M_T = \mathrm{cl}\bigcup\{\mathrm{supp}(m) : m \in \boldsymbol{P}_T\} = \mathrm{cl}\bigcup\{\mathrm{supp}(S'\delta_x) : x \in N^*\}$$

is a P-set (see [8] or [4]). By 3.5, T is strongly ergodic, i.e., $T_n \to R$ in the strong operator topology. By Lemma 3.6 (ii), R is induced by a regular matrix, and, by Lemma 3.6 (i), the convergence is in the uniform operator topology.

5. Remarks. (a) Some readers may wonder what "constructive" meaning is to be assigned to the uniform convergence of the induced operators on $C(N^*)$. Suppose (a_{mn}) is an infinite matrix whose rows are a bounded subset of l^1 and whose columns are null sequences. It defines a bounded operator on $C_b(N)$ which maps c_0 into c_0, and hence induces an operator A on $C(N^*)$. It is easy to check that the norm of A is just $\limsup\limits_{m\to\infty} \sum\limits_n |a_{mn}|$.

(b) There is not very much literature on generalized almost-convergence. For a very general approach and many concrete results, see [7].

REFERENCES

[1] R. Atalla, *On the inclusion of a bounded convergence field in the space of almost convergent sequences*, Glasgow Mathematical Journal 13 (1972), p. 82-90.

[2] — *On the consistency theorem in matrix summability*, Proceedings of the American Mathematical Society 35 (1972), p. 416-422.

[3] — *Generalized averaging operators and matrix summability*, ibidem 38 (1973), p. 372-378.

[4] — *Regular matrices and P-sets in $\beta N \setminus N$*, ibidem 37 (1973), p. 157-162.

[5] — *On the mean convergence of Markov operators*, Proceedings of the Edinburgh Mathematical Society 19 (1974), p. 205-209.

[6] — *Generalized Markov projections and matrix summability*, Canadian Mathematical Bulletin 22 (1979), p. 311-316.

[7] J. Duran, *Almost convergence, summability and ergodicity*, Canadian Journal of Mathematics 26 (1974), p. 372-387.

[8] M. Henriksen and J. Isbell, *Multiplicative summability methods and the Stone-Čech compactification. II*, Notices of the American Mathematical Society 11 (1964), p. 90-91.

[9] B. Jamison and R. Sine, *Sample path convergence of stable Markov process*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 28 (1974), p. 173-177.

[10] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Mathematica 80 (1948), p. 167-190.

[11] W. Rudin, *Functional analysis*, New York 1973.

[12] P. Schaeffer, *Matrix transformations of almost convergent sequences*, Mathematische Zeitschrift 112 (1969), p. 321-325.

[13] R. Sine, *On local uniform mean convergence for Markov operators*, Pacific Journal of Mathematics 60 (1975), p. 247-252.

[14] — *Geometric theory of a single Markov operator*, ibidem 27 (1969), p. 155-166.

DEPARTMENT OF MATHEMATICS
OHIO UNIVERSITY
ATHENS, OHIO

Reçu par la Rédaction le 20. 4. 1979;
en version modifiée le 2. 10. 1979