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SOME RESULTS ON $A_p(G)$ SUBMODULES OF $PM_p(G)$

BY

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ON HIS 70TH BIRTHDAY
WITH RESPECT AND ADMIRATION

Introduction. The present paper contains two somewhat disconnected results.

The first is an analogue of the following result of Glicksberg(1) [8]:

THEOREM. *If G is a compact or a locally compact abelian group then any norm closed left invariant subspace of $L^\infty(G)$, which is reflexive, is necessarily finite dimensional.*

As noted in [8], $L^1(G)$ contains, in the compact abelian case, closed subspaces isomorphic to l^2, spanned by lacunary sets of characters. This is in marked contrast to the above theorem.

In analogy with Glicksberg's result our main result of Section 1 (Theorem 3.1) states the following:

Let G be an amenable locally compact group, $1 < p < \infty$. Let N be a norm closed A_p submodule of $PM_p(G)$, which is a reflexive Banach space. Then N is finite dimensional.

This theorem fails if G is the free group on two generators.

If $p = 2$ and G is abelian then our theorem yields Glicksberg's result. (Since as noted in [8], p. 308, N is necessarily w^* closed. Hence $v \cdot N \subset N$ for all $v \in A_2(G) = (L^1(\hat{G}))\hat{}$ is equivalent to $\chi \cdot N \subset N$ for each $\chi \in \hat{G}$.)

If G is nonabelian then our result is a statement on certain A_p submodules of the algebra of operators $PM_p = A_p^*$ on $L^p(G)$.

Our second main result is a converse of a result of Curtis and Figà-Talamanca [5] which turns out to characterize amenable groups.

(1) May his memory be blessed.

Let G be locally compact abelian and $X \subset L^\infty(G)$ be a norm closed L^1 submodule (i.e., such that $L^1 * X \subset X$). Denote by $\mathcal{M}(X, L^\infty)$ the algebra of all operators T: $X \to L^\infty$ such that $T(f * g) = f * (Tg)$ for all $f \in L^1$, $g \in L^\infty$. If $m \in (L^1 * X)^*$ let $\tau m \in \mathcal{M}(X)$ be defined by $\tau m = m_L$ where $(m_L g)(f) = m(f * g)$ for $f \in L^1$ and $g \in X$. It has been proved by Curtis and Figà-Talamanca [5] that if $X = UC(G) \subset L^\infty(G)$ (the uniformly continuous functions on G) then τ: $UC(G)^* \to \mathcal{M}(L^\infty)$ is an *onto* isometry. This has been improved by A. Lau [13], p. 52, and C. Cecchini [2] to amenable groups G and certain submodules X of $VN(G) = PM_2(G)$. Their proof carries over to any A_p submodules of PM_p.

Our main result in Section 2 is Theorem 2.2.

If $\tau(\overline{A_p \cdot X^*})$ contains the identity injection $I \in \mathcal{M}(X, PM_p)$, given by $I\phi = \phi$ for $\phi \in X$, a fortiori if τ is *onto* $\mathcal{M}(X, PM_p)$, then G is amenable.

In the proof of Theorem 2.1 we use an interesting result of Berg and Christensen [1].

Definitions and notations. For the main part we follow the definitions and notations of our paper [9], most of which are to be found and consistent with Herz [11] and Cowling [4]. For the readers benefit we bring part of them here:

Let G be a locally compact topological group with a fixed left Haar measure $\lambda = dx$ and $L^p(G)$, $1 \leqslant p \leqslant \infty$, the usual complex function spaces on G (see [12]). Let $C(G) \supset UC(G) \supset C_0(G) \supset C_{00}(G)$ be the complex bounded continuous, two-sided uniformly continuous, which tend to 0 at ∞, with compact support, respectively. Let $M(G) = C_0(G)^*$ be the dual of $C_0(G)$. If $\mu \geqslant 0$ then supp μ denotes the smallest closed set $C \subset G$ such that for any open $U \subset G$ such that $U \cap C \neq \emptyset$ we have $\mu(U \cap C) \neq 0$. The spaces $A_p(G)$, $PM_p(G) = A_p(G)^*$, $PF_p(G)$ etc. are as in [9]. We will write $A_p(G) = A_p$, $PM_p(G) = PM_p$ etc. when the context is clear.

If X, Y are Banach spaces over the complex field, X^*, Y^* will denote their Banach space duals, $B(X, Y)$ will denote the bounded linear operators T: $X \to Y$. Operator will always mean a bounded linear operator. We write $B(X, X) = B(X)$.

A subspace will always mean a linear subspace. If Y is a space of linear functionals on X then $\sigma(X, Y)$ will denote the weakest topology on X which makes all linear functionals on X continuous. $\sigma(X, X^*)$, $[\sigma(X^*, X)]$ will stand for the weak [weak*] topology on $X[X^*]$.

If $\mu \in M(G)$ and f, g are measurable functions on G then the convolutions $\mu * g$, $f * g$ will be as in Hewitt and Ross [12]. These are consistent with [4], [9], [11].

If A is a subset of the Banach space X then $\bar{A}$ will denote the norm closure of A.

We note the module action of A_p on PM_p or PM_p^* given by $\langle u\cdot\phi, v\rangle = \langle\phi, uv\rangle$, $\langle u\cdot m, \phi\rangle = \langle m, u\cdot\phi\rangle$ for $u, v\in A_p$, $\phi\in PM_p$, $m\in PM_p^*$.

1. Reflexive invariant subspaces of $PM_p(G)$. Recently, Glicksberg [8] proved that if G is a compact or abelian group and N is a norm closed translation invariant reflexive subspace of $L_\infty(G)$, then N is finite dimensional. In this section, we shall prove an analogue of Glicksberg's result for subspaces of $PM_p(G)$.

As usual denote by $M(G)$ the space of bounded regular Borel measures on G, $1<p<\infty$. If $f\in L^1(G)$, $\mu\in M(G)$ let $\varrho(\mu)$, $\varrho(f)$ be the operators on $L^p(G)$ defined by: $\varrho(f)(h) = f*h$, $\varrho(\mu)(h) = \mu*h$. Let $PF_p(G)$ be the norm closure of $\{\varrho(f)\colon f\in L_1(G)\}$ in $B(L_p(G))$. Then $PF_p(G)\subset PM_p(G) = A_p(G)^*$ and $PF_p(G)$ is weak*-dense in $PM_p(G)$. The dual space $PF_p(G)^*$ has been identified with a subalgebra of $C(G)$ by Cowling in [4]. If G is amenable, then $PF_p(G)^* = B_p(G)$, the algebra of pointwise multipliers on $A_p(G)$. In this case $1\in PF_p(G)^*$. (See Cowling [4], p. 91, for details on the above.)

If one identifies $\mu\in M(G)$ with $\varrho(\mu)$ then $M(G)\subset PM_p(G) = A_p(G)^*$ with duality expressed by $\langle\mu, h\rangle = \int h(x)\,d\mu(x)$ for $h\in A_p$.

LEMMA 1.1. *Let G be a locally compact group, $1<p<\infty$ and let $h\in PF_p(G)^*$. Then $\langle h, \mu\rangle = \int h(x)\,d\mu(x)$ is a continuous linear functional on the normed linear space $(M(G), \|\cdot\|_{PM_p})$.*

Proof. Let $\{V_\alpha\}$ be a base of neighborhoods of the identity e of G such that each V_α has compact closure. Let $k_{V_\alpha}\in C(G)$, $k_{V_\alpha}\geqslant 0$ and with support contained in V_α and $\int k_{V_\alpha}(x)\,dx = 1$. Let $\mu\in M(G)$. Define $f_{V_\alpha} = \mu*k_{V_\alpha}$. Then $f_{V_\alpha}\in L_1(G)$, and as pointed out in Eymard [7, p. 190], for each $g\in C(G)$, the net $\int f_{V_\alpha}(x)g(x)\,dx$ converges to $\int g(x)\,d\mu(x)$. Since $h\in C(G)$, we have

$$\Big|\int h(x)\,d\mu(x)\Big| = \lim_\alpha\Big|\int f_{V_\alpha}(x)h(x)\,dx\Big| \leqslant \|h\|_{PF_p^*}\|f_{V_\alpha}\|_{PF_p}$$

$$\leqslant \|h\|_{PF_p^*}\|\mu\|_{PM_p}\|k_{V_\alpha}\|_{PF_p}$$

$$\leqslant \|h\|_{PF_p^*}\|\mu\|_{PM_p}$$

since $\|k_{V_\alpha}\|_{PF_p}\leqslant\|k_{V_\alpha}\|_{L_1} = 1$.

If $x\in G$, write $\lambda(x) = \varrho(\delta_x)$, where $\delta_x\in M(G)$ is the point mass at x.

LEMMA 1.2. *Let G be an amenable locally compact group. Let $S\subset G$ such that $\{\lambda(x);\ x\in S\}$ is a weakly (i.e. $\sigma(PM_p, PM_p^*)$) relatively compact subset of $PM_p(G)$. Then S is finite.*

Proof. Let C be the weak closure of $\{\lambda(x);\ x\in S\}$ in $PM_p(G)$. If C is weakly compact and S is infinite, then we can by Smulian's theorem assume that there exists a sequence $\{x_n\}$ of distinct elements in S such that $\lambda(x_n)$ converges to T *weakly*, for some $T\in PM_p(G)$, as $n\to\infty$. In particular, $\lambda(x_n)$

converges to T in the weak*-topology of $PM_p(G)$ also. Since each $\lambda(x_n)$ is a multiplicative linear functional on $A_p(G)$, T is also a multiplicative linear functional on $A_p(G)$, or $T = 0$. Thus $T = \lambda(x_0)$ for some $x_0 \in G$, or $T = 0$ [11, p. 102]. Now $\{\lambda(x_n);\ n \geqslant 1\} \cup \{T\}$ is weakly compact. Hence the weak and the weak*-topology coincide on it.

If $T = \lambda(x_0)$, then $x_0 \neq x_n$ for each $n = 1, 2, 3, \ldots$, since we assume that the x_n's are distinct. Let m be a topological invariant mean on $PM_p(G)$, i.e. $m \in PM_p^*(G)$; $\|m\| = m(I) = 1$, $u \cdot m = u(e)m$ for all $u \in B_p(G)$, where $\langle u \cdot m, T\rangle = \langle m, u \cdot T\rangle$ and $\langle u \cdot T, v\rangle = \langle T, uv\rangle$ for all $v \in A_p(G)$, $T \in PM_p(G)$. That such m exists is proved in [9, Theorem 5]. Also, $m(\mu) = \mu\{e\}$ for each $\mu \in M(G)$ [9, Proposition 10]. For $b \in G$, let $m_b \in PM_p(G)^*$ be defined by $m_b(T) = m(\lambda(b)T)$, $T \in PM_p(G)$. Then

$$m_b(\lambda(a)) = m(\lambda(b)\lambda(a))$$

which is one when $ba = e$ and zero when $ba \neq e$. Hence $m_{x_0^{-1}}(\lambda(x_n)) = 0$ since $x_0^{-1}x_n \neq e$ for each n. But $m_{x_0^{-1}}(T) = m_{x_0^{-1}}(\lambda(x)) = 1$. Hence the sequence $\{\lambda(x_n)\}$ cannot converge to $\lambda(x_0)$ weakly.

If $T = 0$, then by Lemma 3.1 and the amenability of G one has $1 \in PF_{p^*}$ (see Cowling [4]); hence $\mu \to \langle 1, \mu\rangle$ is a continuous linear functional on the normed linear space $(M(G), \|\cdot\|_{PM_p})$. But then $1 = \langle 1, \lambda(x_n)\rangle \to \langle 1, T\rangle = 0$ which cannot be.

THEOREM 1.3. *Let G be an amenable locally compact group, $1 < p < \infty$. Let N be a norm closed, A_p submodule of $PM_p(G)$ which is a reflexive Banach space. Then N is the linear span of $\{\lambda(a_i);\ 1 \leqslant i \leqslant n\}$ for some finite set $\{a_1, \ldots, a_n\} \subset G$.*

Proof. Let B be the closed unit ball of N. Then B is $\sigma(PM_p, PM_p^*)$ (i.e. weakly) compact. Hence the weak and $w^* = \sigma(PM_p, A_p)$ topologies coincide on B and B is also w^* compact. By the Krein–Smulian theorem (see [6, p. 429]) N is w^* closed in $PM_p(G)$ (until here our proof follows Glicksberg's proof in [8]). Let $S = \{\lambda(x);\ \lambda(x) \in N\}$. Then $S \subset B$, hence S is weakly relatively compact. Thus S is finite by Lemma 1.2. Let $S = \{\lambda(a_k);\ k = 1, 2, \ldots, n\}$ where $a_k \in G$. We claim that N coincides with the linear span of S. In fact if $T \in N$ then for any $a \in \operatorname{supp} T$ there is a net $v_\alpha \in A_p$ such that $v_\alpha \cdot T \to \lambda(a)$ in w^* (see [11], p. 101). But then $\operatorname{supp} T \subset S$. Let hence $\operatorname{supp}(T) = \{a_k\colon 1 \leqslant k \leqslant l\}$. Let V be a neighborhood of e such that $\bar{V}$ is compact and $a_k V^3 \cap a_j V^3 = \emptyset$ if $k \neq j$. Let $v \in A_p$ be such that $\|v\| = 1$, $v(V) = 1$, and $v = 0$ off V^2 (see [9]). Let $v_i = v_{a_i^{-1}}$ where $v_a(x) = v(ax)$. Then by Herz [11], p. 118, supp $v_i \cdot T = \{a_i\}$ for $k \leqslant l$. Thus $v_i \cdot T = \alpha_i \lambda(a_i)$ for some scalars α_i. Thus $(\sum v_i) \cdot T = \sum \alpha_i \lambda(a_i)$. But $u = \sum^{l} v_i \in A_p$ and $u = 1$ on a neighborhood of the support of T. Thus $uT = T = \sum_1 \alpha_i \lambda(a_i)$.

Remark. The following is an example of a nonamenable discrete group for which both Theorem 1.3 and Lemma 1.2 do not hold:

Let G be a discrete group which contains a free group on two generators. Then G contains an infinite Leinert set E by M. Leinert, *Multiplikatoren gewisser diskreter Gruppen*, Studia Math. 52 (1974), pp. 149–158, which satisfies:

If $N = \{f \in PM_2;\ \operatorname{supp} f \subset E\}$ then $N = l^2(E)$ and there is some $C > 0$ such that $\|f\|_{PM_2} \leqslant C\|f\|_{l^2}$.

It follows easily that N is an *infinite dimensional* norm closed A_2-submodule of PM_2 on which the l^2 norm and the PM_2 norm are equivalent. In particular N is a *reflexive* Banach space. Furthermore N is w^* closed. Let $S = \{\lambda(x);\ x \in E\}$. Then S is an *infinite* subset of the unit ball of N (which is weakly i.e. $\sigma(PM_2, PM_2^*)$ compact) and is hence weakly relatively compact, i.e. whose weak closure is weakly compact. Hence Theorem 1.3 and Lemma 1.2 fail for the nonamenable group G.

The reader will no doubt note that if $G = R$ is the real line then $PM_2(R) = L^\infty(R)$ and *every* separable (reflexive) Banach space is isometric to a subspace of $L^\infty(R)$.

2. A_p submodules of PM_p and amenability.

Definition. The linear subspace $X \subset PM_p$ is called an A_p *submodule* (a topologically invariant subspace in Lau [13]) if $A_p \cdot X \subset X$. Let $X \subset PM_p$ be an A_p submodule. For each $m \in X^*$ define the operator m_L: $X \to PM_p$ by $(m_L\phi)(v) = m(v\cdot\phi)$ for v in A_p. Note that m need only be in $(A_p\cdot X)^*$ for m_L to make sense.

Clearly $\|m_L\| \leqslant \|m\|$ and $m_L(u\cdot\phi) = u\cdot m_L(\phi)$ for each $\phi \in X$, and $u \in A_p$ and $m \in X^*$. Furthermore $m_L = 0$ iff $m(A_p\cdot X) = 0$. Hence the map τ: $\overline{(A_p\cdot X)}^* \to B(X, PM_p)$ given by $\tau(m) = m_L$ is one-to-one and satisfies $u\cdot[\tau m\phi] = \tau(u\cdot m)\phi$, where $(u\cdot m)(\phi) = m(u\cdot\phi)$ for $u \in A_p$, $m \in \overline{(A_p\cdot X)}^*$, $\phi \in X$.

The A_p submodule $X \subset PM_p$ is called *topologically introverted* if $m_L X \subset X$ for all $m \in X^*$. For example PM_p, UC_p, PF_p are such (see Lau [13], p. 48, for $p = 2$).

If X, Y are A_p submodules of PM_p, then we write

$$\mathcal{M}(X, Y) = \{T \in B(X, Y);\ T(u\cdot\phi) = u\cdot T(\phi) \text{ for all } u \in A, \phi \in X\}.$$

Let $\mathcal{M}(X) = \mathcal{M}(X, X)$. Clearly the identity operator I: $X \to X$ belongs to $\mathcal{M}(X)$. If $X \subset Y$ then the injection I: $X \to Y$ ($I\phi = \phi$) belongs to $\mathcal{M}(X, Y)$.

It has been proved by Curtis and Figà-Talamanca [5] that if G is locally compact *abelian* then τ is an isometry from $UC(G)^* = UC_2(G)^*$ onto $\mathcal{M}(PM_2)$ ($PM_2 = L^\infty(\hat{G})$ if G is abelian). This has been improved to all amenable G and $p = 2$ by Lau in [13], p. 52, and independently by Carlo Cecchini [2]. We improve their result in the next Proposition 2.1. We find it of interest that this Curtis–Figà-Talamanca property (in fact a weaker property than it) turns out to be a characterization of amenability for G. This seems to be new even for $p = 2$. This is proved in the next Theorem 2.2, the main result of this section. Write $UC_p = \overline{A_p \cdot PM_p}$.

PROPOSITION 2.1. *Let X be a norm closed A_p-submodule of PM_p. Then*

(a) $\tau\colon (\overline{A_p \cdot X})^* \to \mathcal{M}(X, PM_p)$ *defined by* $\tau m = m_L$ *is a norm decreasing injection such that* $u \cdot \tau(m)\phi = \tau(u \cdot m)\phi$ *for all* $u \in A_p$ *and* $m \in (\overline{A_p \cdot X})^*$, $\phi \in X$. *If X is introverted, τ is into* $\mathcal{M}(X)$.

(b) *If G is amenable then τ is an isometry onto. If in addition X is topologically introverted and* $X \subset UC_p$ *then*

$$\tau(\overline{A_p X})^* = \mathcal{M}(X), \quad [\tau(X^*) = \mathcal{M}(X)].$$

Proof. (a) is proved in the above remarks.

(b) This part improves Theorem 6.2 of [13], p. 52:

If G is amenable let v_n be an approximate identity in A_p such that $\|v_n\| \leqslant 1$. If $\phi \in X$ and $m \in (\overline{A_p \cdot X})^*$ then $\|m_L \phi\| \geqslant |(m_L \phi) v_n| = |m(v_n \cdot \phi)| \to |m(\phi)|$ if $\phi \in \overline{A_p \cdot X}$. If $\phi_0 \in A_p \cdot X$, $\|\phi_0\| = 1$ and $m(\phi_0) \geqslant \|m\| - \varepsilon$ then $\|m_L \phi_0\| \geqslant \|m\| - \varepsilon$. Thus $\|m_L\| = \|m\|$ and $A_p \cdot X$ is even a *norming* subspace for $m_L\colon X \to PM_p$. Let now $T \in \mathcal{M}(X, PM_p)$. Then $v_n \in A_p \subset PM_p^*$ and $\|T^* v_n\| \leqslant \|T^*\|$. Let $m = w^* \lim_\alpha T^* v_{n_\alpha}$ in $(\overline{A_p \cdot X})^*$. Then for each $v \in A_p$, $\phi \in X$:

$$\begin{aligned}\langle T\phi, v\rangle &= \lim_\alpha \langle T\phi, v v_{n_\alpha}\rangle = \lim_\alpha \langle vT\phi, v_{n_\alpha}\rangle = \lim_\alpha \langle Tv \cdot \phi, v_{n_\alpha}\rangle \\ &= \lim_\alpha \langle v \cdot \phi, T^* v_{n_\alpha}\rangle = m(v \cdot \phi) = m_L \phi(v).\end{aligned}$$

It follows that $T = m_L$ and that $\tau(\overline{A_p \cdot X})^* = \mathcal{M}(X, PM_p)$. If X is introverted then by definition $m_L X \subset X$ for all $m \in (\overline{A_p \cdot X})^*$. Thus $\tau[(\overline{A_p \cdot X})^*] = \mathcal{M}(X)$.

[If $X \subset UC_p$ then $\overline{A_p \cdot X} = X$ by a routine argument (see Lau [13], p. 53). Thus $(\overline{A_p \cdot X})^* = X^*$, in this case.]

THEOREM 2.2. *Let X be a norm closed A_p submodule of PM_p which either*

(a) *contains PF_p (PF_p, W_p, UC_p, PM_p, e.g. are such) or*

(b) *contains a probability measure μ_0 such that* $e \in \operatorname{supp}(\mu_0)$, $\operatorname{supp}(\mu_0)$ *generates a dense subgroup of G and the convolution powers μ_0^{*n} belong to X for all $n \geqslant 1$.*

If $\tau(\overline{A_p \cdot X})^*$ contains the identity injection $I \in \mathcal{M}(X, PM_p)$ (a fortiori if $\tau(\overline{A_p \cdot X})^* = \mathcal{M}(X, PM_p)$) then G is amenable.

Remark. It is enough in (b) that $\mu_0^{*n_k}$ belongs to X for an infinite sequence $\{n_k\}$ of positive integers.

Proof. Let $m \in X^*$ satisfy $m_L = I\colon X \to PM_p$. Then in case (a), for each $f \in L^1(G)$ and $v \in A_p$ we have $(m_L f)(v) = m(v \cdot f) = (If)(v) = \int fv\, dx$. Hence

$$(*) \qquad \left|\int f\, v\, dx\right| \leqslant C\, \|v \cdot f\|_{PM_p},$$

where $C = \|m\|$. Let G_0 be open and σ-compact and $f = 0$ a.e. outside G_0. Let $K_n \subset K_{n+1}$ be compact such that $\bigcup_n K_n = G_0$. Let $v_n \in A_p \cap C_{00}(G)$ be such that $v_n(K_n) = 1$ and $0 \leqslant v_n \leqslant 1$. Then $\|v_n \cdot f - f\|_{L^1} = \int |v_n f - f|\, dx \to 0$. It follows that $\|v_n \cdot f - f\|_{PM_p} \to 0$ and $\int v_n f\, dx \to \int f\, dx$. (Note that $\sup_n \|v_n\|_{A_p} = \infty$ may hold.) Applying $(*)$ we get that $\left|\int f\, dx\right| \leqslant C\|f\|_{PM_p}$ for all f in $L^1(G)$. Taking now convolution powers we get for all $n \geqslant 1$ that:

$$\left|\int f\, dx\right|^n = \left|\int f^{*n}\, dx\right| \leqslant C\, \|f^{*n}\|_{PM_p}.$$

Hence $\left|\int f\, dx\right| \leqslant C^{1/n} \|f^{*n}\|_{PM_p}^{1/n} \to |\varrho(f)|$ as $n \to \infty$ where $|\varrho(f)|$ denotes the spectral radius of the convolution operator $\varrho(f)\colon L^p(G) \to L^p(G)$ given by $\varrho(f)(g) = f * g$. Hence, if $f \geqslant 0$ is in $L^1(G)$ then $\|f\|_{L^1} = |\varrho(f)|$. But this is precisely condition E_p of Leptin, which implies the amenability of G, by Leptin [14], p. 493, Theorem 1.

In case (b) we will have $(m_L \mu)(v) = \mu(v) = \int v\, d\mu$ hence $(*)$ will be replaced by $\left|\int v\, d\mu\right| \leqslant C\, \|v \cdot \mu\|_{PM_p}$ for any bounded Borel measure $\mu \in X$. Choose G_0 and the functions v_n for a fixed measure $\mu \in X$ as those for $f \in L^1(G)$ above. Then $\|v_n \cdot \mu - \mu\|_{PM_p} \leqslant \|v_n \cdot \mu - \mu\|_{M(G)} \to 0$. We get as above that for each bounded measure μ in X we have $|\mu(G)| \leqslant C\, \|\mu\|_{PM_p}$. In particular this holds for μ_0^{*n} for all n. Thus

$$|\mu_0(G)|^n = |\mu_0^{*n}(G)| \leqslant C\, \|\mu_0^{*n}\|_{PM_p}.$$

This implies that

$$1 = \mu_0(G) \leqslant C^{1/n} \|\mu_0^{*n}\|^{1/n} \to |\varrho(\mu_0)| \leqslant \|\mu_0\|_{PM_p} \leqslant 1$$

where $|\varrho(\mu_0)|$ is the spectral radius of the operator $\varrho(\mu_0)(g) = \mu_0 * g$ on $L^p(G)$. Hence $\|\mu_0\|_{PM_p} = |\varrho(\mu_0)| = 1$. Apply now Theorem 1 of Berg and Christensen [1] and get that G is amenable. This finishes the proof.

Remarks. 1. We only needed the fact that $\mu_0^{*n_k} \in X$ for some sequence $n_k \to \infty$.

2. Assume that G is second countable and consider $AP_p(G)$, the almost periodic elements of PM_p (see [9]). It is not known if AP_p is a subalgebra of PM_p. It is however an A_p submodule of PM_p (see [9], Section II). Furthermore it is not clear if $AP_p \subset UC_p$ for nonamenable G. Assume though that $\tau(\overline{A_p \cdot AP_p})^*$ contains the identity I. Does this force G to be

amenable? The answer is *yes*: Since let $\mu_0 = \sum_0^\infty 2^{-n-1}\delta_{x_n}$ where $e = x_0$ is the unit of G and $\{x_n;\ n \geqslant 0\}$ are dense in G. By Proposition 12 of [9], $l^1(G) \subset AP_p$, and a fortiori, μ_0^{*n} belongs to $AP_p(G)$ for all n. Part (b) of the above theorem will finish the proof.

3. Berg and Christensen note in Remark 3 on p. 152 of [1] that if μ_0 is a probability measure on G such that $e \in \operatorname{supp}\mu_0$ and such that $\|\mu_0\|_{PM_p} = 1$ for some $1 < p < \infty$ then $\|\mu_0\|_{PM_2} = 1$ (by p. 150) and $LUC(G)$, the bounded left uniformly continuous functions on G, admit a G_0-left invariant mean, where G_0 is the closed subgroup generated by $\operatorname{supp}\{\mu_0\}$. Using this fact one can show that if the norm closed A_p submodule X contains $l^1(G)$ and if $\tau(A_p \cdot X)^*$ contains the identity $I \in \mathcal{M}(X, PM_p)$ then G is amenable. One proves this fact by using the net of all separable closed subgroups $G_\alpha \subset G$ directed by inclusion. One shows then that $LUC(G)$ admits for each α some G_α left invariant mean. But then any function $F = \sum_1^n (f_i - l_{a_i} f_i)$ with $a_i \in G$, $f_i \in LUC(G)$ satisfies $\sup_x F(x) \geqslant 0$. This implies that G is amenable. It follows that even if G is not separable our theorem applies for $X = AP_p$.

4. Assume that G is not amenable and X is such that for some compact $K \subset G$, $\operatorname{supp}\phi \subset K$ for all ϕ in X. If $v \in A_p \cap C_{00}$ is such that $v = 1$ on some neighborhood of K then consider v as belonging to PM_p^*. Clearly $v_L = I \in \mathcal{M}(X, PM_p)$, yet G is not amenable.

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