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ON UNIQUELY ARCWISE CONNECTED CURVES

BY

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Introduction. A topological space X has the fixed point property if for every continuous transformation f: $X \to X$ there exists a fixed point, i.e. a point $x \in X$ such that $x = f(x)$ (see [2] for an expository article). The investigation of the fixed point property of continua of the lowest dimension, called *curves*, can be restricted to those curves X which do not contain any homeomorphic image of the unit circle, called a *simple closed curve* – because such a simple closed curve would be a retract of X (see [9], vol. II, p. 354, Theorem 1). X is *arcwise connected* if every pair of points of X can be joined in X by a homeomorphic image of the unit segment of the real line, called an *arc*. Then the condition that X does not contain any simple closed curve is equivalent to saying that X contains a unique arc between an arbitrary pair of points, i.e. that X is uniquely arcwise connected.

This property of a curve X does not characterize the fixed point property of X as was shown by Young [20] (see also [17] for another example). Thus the main task is to determine the class of these uniquely arcwise connected curves which have the fixed point property. The most important contribution to this problem is given by Hagopian [6] and Mohler [15]. However, those two results seem not to give a general method for studying uniquely arcwise connected curves.

The main theorem of the present paper, Theorem 2, is supposed to be the first step for such a general study, formalizing the so called *dead-end method* by [2] (p. 123–125). The proof of Theorem 2 is in its first part similar to the proof of Zermelo–Hessenberg fixed point theorem [4] which is given e.g. in [16]. This shows another connection between set theory and the fixed point property of curves (see [13] for a relationship between Zermelo–Hessenberg and Knaster–Tarski theorem, [8]). Theorem 2 is used here to prove Theorem 3 and state a property of the class $\mathscr{X}$ of arcwise connected curves having the fixed point property, namely that X, Y, $X \cap Y \in \mathscr{X}$ do not imply $X \cup Y \in \mathscr{X}$ (see Corollary 4).

Theorem 1 plays an auxiliary role in the paper, but it gives a new characterization of a simple closed curve (Corollary 1) and seems to be related to Problem 1.

The main Theorem 2 was inspired by an unpublished result of K. Sieklucki, quoted here as Corollary 2. The author wishes to express his gratitude to Professor K. Sieklucki for encouragement to elaborate this general theorem in the paper. The paper has been written during the author's work at Seminar on Geometric Topology at the Warsaw University.

1. One-to-one continuous images of $[0, \infty)$. In this section we shall consider metric spaces of the form $P = \varphi([0, \infty))$ where φ is one-to-one transformation of the real half-line $[0, \infty)$. Then P will be called a *ray* and φ will be always assumed to be a one-to-one continuous transformation.

LEMMA 1. *A ray $P = \varphi([0, \infty))$ contains a simple closed curve C if and only if there exists $t_1 \in [0, \infty)$ such that the closure $\overline{\varphi([t_1, \infty))}$ in P is an arc. In such a case $C = \overline{\varphi([t_0, \infty))}$ for some $t_0 < t_1$.*

Proof. *Sufficiency.* Let I denote the arc $\overline{\varphi([t_1, \infty))}$ in P. Since $\varphi([t_1, \infty))$ is the union of a strongly increasing sequence of arcs $\varphi([t_1, t_n]) \subset I$, where $t_1 < t_2 < \ldots$, we infer that $\varphi([t_1, \infty))$ is a half-open arc in I. Therefore $\overline{\varphi([t_1, \infty))} - \varphi([t_1, \infty)) = \{\varphi(t_0)\}$ for some $t_0 < t_1$. It follows that two arcs $\varphi([t_0, t_1])$ and $\overline{\varphi([t_1, \infty))}$ have only the end points $\varphi(t_0)$ and $\varphi(t_1)$ in common. Hence the union $\varphi([t_0, t_1]) \cup \varphi([t_1, \infty))$ is a simple closed curve.

Necessity. Let C be a simple closed curve contained in P. It will be shown that $C = \varphi([t_0, \infty))$ for some $t_0 \in [0, \infty)$.

Namely, if there were an unbounded sequence $t_1 < t_2 < \ldots$ such that $\varphi(t_n) \in P - C$ for all $n = 1, 2, \ldots$, then C would be contained in some arc $\varphi([t_n, t_{n+1}])$ by a theorem of Sierpiński (see [9], p. 173, Theorem 6). Thus $\varphi([t, \infty)) \subset C$ for some $t \in [0, \infty)$. This inclusion holds also for the infimum t_0 of all such t's by the continuity of φ, and then

$$\varphi([t_0, \infty)) \subset C, \tag{1.1}$$

$$\varphi([s, t_0]) - C \neq \emptyset \quad \text{for each } s \in [0, t_0). \tag{1.2}$$

Suppose now, on the contrary, that $\varphi([t_0, \infty)) \neq C$. Then by (1.1), $\varphi([t_0, \infty))$ is a half-open arc in C with the end point $\varphi(t_0)$. Let J denote any other arc of C having only the end point $\varphi(t_0)$ in common with $\varphi([t_0, \infty))$, the other end point of J being $\varphi(s)$ for some $s < t_0$. Then

$$J \subset C, \tag{1.3}$$

$$J \subset \varphi([0, t_0]). \tag{1.4}$$

From (1.2) and (1.3) it follows that the arcs J and $\varphi([s, t_0])$ are different from each other and they have the same end points $\varphi(s)$ and $\varphi(t_0)$. Therefore the union $J \cup \varphi([s, t_0])$ contains a simple closed curve, contained in the arc $\varphi([0, t_0])$ in view of (1.4), which is impossible.

THEOREM 1. *A ray $P = \varphi([0, \infty))$ has the fixed point property for. continuous transformations of P onto itself if and only if P is uniquely arcwise connected.*

Proof. *Sufficiency.* If P is uniquely arcwise connected, then P does not contain any simple closed curve.

Then for every continuous transformation $f\colon P \to P$ and for every $t \geqslant 0$ there exists $\tau \geqslant t$ such that $f(\varphi([0, t])) \subset \varphi([0, \tau])$.

Indeed, the continuum $K = f(\varphi([0, t]))$ is a dendrite. Since, by Lemma 1, the closure $\overline{\varphi([n, \infty))}$ is not an arc for each $n = 1, 2, \ldots$, we have $\varphi([n, \infty)) - K \neq \emptyset$ (see [14], p. 125, Lemme 11; also the proof of Corollary 3 below). Thus there exists a sequence $t_1 < t_2 < \ldots$ such that $\varphi(t_n) \notin K$ for all $n = 1, 2, \ldots$ and by a theorem of Sierpiński (see [9], p. 173, Theorem 6), K is contained in some arc $\varphi([t_n, t_{n+1}])$.

If moreover $f(P) = P$ (by assumption), then there exists $t \geqslant 0$ such that $f(\varphi(t)) = \varphi(0)$ and, as is proved above, the transformation $\hat{f}\colon [0, t] \to [0, \infty)$ defined by the following formula

$$\hat{f}(s) = \varphi^{-1}\big(f(\varphi(s))\big) \quad \text{for all } s \in [0, t]$$

is continuous. Of course, $0 \leqslant \hat{f}(0)$ and $0 = \hat{f}(t) \leqslant t$ by the choice of t. Therefore, there exists a fixed point s_0 of $\hat{f}$ (e.g. by an argument using the diagonal of the square $[0,\infty) \times [0, \infty)$). Then $\varphi(s_0)$ is a fixed point of f.

Necessity. If, on the contrary, P is not uniquely arcwise connected, i.e. if P contains a simple closed curve C, then by Lemma 1, $P = \varphi([0, t_0]) \cup C$ and $C \cap \varphi([0, t_0]) = \{\varphi(t_0)\}$ for some $t_0 \in [0, \infty)$. Thus, there exists a continuous transformation f of P onto itself which has no fixed point.

COROLLARY 1. *A ray P is a simple closed curve if and only if P does not have the fixed point property for homeomorphisms (or one-to-one continuous transformations) of P onto itself.*

Remark 1. (a) In view of Lemma 1, Theorem 1 is a direct generalization of Proposition 1 of [19]. But also, conversely, that proposition can be used instead of Sierpiński's theorem to prove the sufficiency in Theorem 1.

(b) As was observed by Mark Marsh, the "onto" in Theorem 1 cannot be omitted for the rays which are not homeomorphic to $[0, \infty)$. Namely, let $\varphi\colon [0, \infty) \to S^1 \times S^1$ be defined as $\varphi(t) = (e^{2\pi i t}, e^{2\sqrt{2}\pi i t})$ and let $P = \varphi([0, \infty))$. Then the transformation $f\colon P \to P$ given by the formula $f(p, q) = (p, e^{2\sqrt{2}\pi i} q)$ is continuous and has no fixed point.

(c) From Bellamy's example [1], we can derive a planar ray which is not homeomorphic to $[0, \infty)$ and does not have the fixed point property (for continuous transformations into).

(d) Lemma 1 and Theorem 1 make use of some ideas of [7] and [10]. I would like to thank the referee for this information.

PROBLEM 1 (cf. [12], p. 434–435). Has the rational composant (with the end point) of the simplest Knaster indecomposable continuum the fixed point property? **(P 1296)**

2. Uniquely arcwise connected continua. In this section, X will always denote an arbitrary uniquely arcwise connected metric continuum. Then the common part of any arcwise connected subcontinua of X is an arcwise connected subcontinuum of X. For every two arcwise connected subcontinua $K, L \subset X$ such that $K \cap L = \emptyset$ there exists the smallest arc joining these continua in X, i.e. a unique arc $[x, y] \subset X$ such that $K \cap [x, y] = \{x\}$ and $[x, y] \cap L = \{y\}$. It follows that $[x, y] \cup [y, z] \subset X$ is an arc $[x, z] \subset X$ if and only if $[x, y] \cap [y, z] = \{y\}$. Moreover, this condition is equivalent to each of the following: $[x, y] \subset [x, z]$ or $y \in [x, z]$.

For an arbitrary point $x_0 \in X$, a partial order $\leqslant$ is determined in X by the following formula

$$x \leqslant y \quad \text{iff} \quad [x_0, x] \subset [x_0, y]$$

and, moreover, if $x \neq y$, then we write $x < y$.

For every ray $P \subset X$, where $P = \varphi([0, \infty))$ with a one-to-one continuous function φ, let $x_n = \varphi(n)$ and $P(x_n) = \varphi([n, \infty))$ for each $n = 1, 2, \ldots$, i.e.

$$P(x_n) = \{x \in P\colon\ x_n \leqslant x\}.$$

Then the closures $\overline{P(x_n)}$ in X form a decreasing sequence of non-empty continua, and hence the set $L(P) = \bigcap_{n=1}^{\infty} \overline{P(x_n)}$ is a non-empty continuum in X. Equivalently, $L(P)$ is the set of all limit points of P by [3]. If $Q(x_n)$ denote the arc component of $X - \{x_n\}$ which contains $P(x_n) - \{x_n\}$ and if $\bigcap_{n=1}^{\infty} \overline{Q(x_n)} = L(P)$, then P will be said to have small branches. P will be said to *have no branches* if there exists n_0 such that $Q(x_n) \cup \{x_n\} = P(x_n)$ for all $n \geqslant n_0$.

LEMMA 2. *A ray P in a uniquely arcwise connected continuum X has the continuum $L(P)$ consisting of one point if and only if P is contained in an arc in X.*

Proof. If $L(P)$ consists of one point $y_0 \in X$, then $\bar{P} = P \cup \{y_0\}$. Suppose, on the contrary, that $y_0 \in P$ and take $x \in P$ such that $y_0 \notin P(x)$. Then, of course being $L(P(x)) = \{y_0\}$, $\overline{P(x)} \cap P = P(x) \cup \{y_0\}$ is an arc. It follows by Lemma 1 that P contains a simple closed curve, contradicting the assumption on X.

THEOREM 2. *Let $f\colon X \to X$ be a continuous transformation of a uniquely arcwise connected continuum X and choose $x_0 \in X$ to define the order $\leqslant$ in X. If $x_0 \neq f(x_0)$, then there exists a unique ray $P \subset X$ with the initial point x_0 which is maximal with respect to*

$$(2.1) \qquad x < f(x) \quad \textit{for all } x \in P.$$

Then $L(P) \subset f(L(P))$, and the converse inclusion holds too (for $L(P)$ non-degenerated to one point) if P has small branches.

Proof. Consider the family $\mathscr{P}_0$ of all rays $P_\alpha = [x_0, x_\alpha)$ in X, $\alpha \in \mathscr{A}$, which satisfy condition (2.1) with P_α instead of P.

The family $\mathscr{P}_0$ is monotone, i.e. for each $P_\alpha, P_{\alpha'} \in \mathscr{P}_0$, $P_\alpha \subset P_{\alpha'}$ or $P_{\alpha'} \subset P_\alpha$. To see this, it is sufficient to observe that for every $P_\alpha \in \mathscr{P}_0$ the condition (2.1) implies that for every $x \in P$ there exists $x' \in P_\alpha$ such that $x < x'$ and that the arc $[x, x']$ is contained in the set $[x, x_\alpha] \cap [x, f(x)]$ which itself is an arc with one end point x (by the uniquely arcwise connectedness of X), so that $[x, x_\alpha] \cap [x, f(x)]$ is an arc non-degenerated to the point x; then it will be written $[x, x_\alpha] -\!\!< [x, f(x)]$ (cf. [11]).

Thus it is to be proved that (2.1) implies

$$(2.2) \qquad [x, y] -\!\!< [x, f(x)] \quad \text{for each } x, y \in P \text{ with } x < y.$$

Indeed, let $x, y \in P$ and $x < y$. By (2.1), $f(x) \notin [x_0, x]$ and hence by the continuity of f there exists $x' \in X$ such that

$$x < x' < y$$

and $f([x, x']) \cap [x_0, x'] = \emptyset$. Consider the smallest arc I joining in X these disjoint arcwise connected continua. This arc I is contained in $[x, f(x)]$ and if, on the contrary, $[x, y] \cap [x, f(x)] = \{x\}$, then $I = [x, z]$ for some $z \in f([x, x'])$ so that $[x_0, x'] \cap [x, z] = \{x\}$. Then

$$x' \notin [x_0, x] \cup [x, z] \cup [z, f(x')]$$

and since this union of arcs contains the arc $[x_0, f(x')]$, it follows that $x' \notin [x_0, f(x')]$ contradicting (2.1).

The family $\mathscr{P}_0$ is non-empty. Since $x_0 \neq f(x_0)$ by assumption, hence, by the continuity of f, there exists an arc $[x_0, x_1] \subset [x_0, f(x_0)]$ such that

$$(2.3) \qquad [x_0, x_1] \cap f([x_0, x_1]) = \emptyset.$$

In the arc $[x_1, f(x_0)]$ take a subarc $[x_1, y_1]$ joining $[x_0, x_1]$ and $f([x_0, x_1])$ so that

$$(2.4) \quad [x_0, x_1] \cap [x_1, y_1] = \{x_1\} \quad \text{and} \quad [x_1, y_1] \cap f([x_0, x_1]) = \{y_1\}.$$

It will be now proved that $[x_0, x_1) \in \mathscr{P}_0$.

Let $x \in [x_0, x_1)$. Then $[y_1, f(x)] \subset f([x_0, x_1])$, hence, by (2.4),

$[x_1, y_1] \cap [y_1, f(x)] = \{y_1\}$, and therefore $[x_1, y_1] \cup [y_1, f(x)] = [x_1, f(x)]$. Hence $[x_0, x_1] \cap [x_1, f(x)] = \{x_1\}$ by (2.3) and (2.4), i.e. $[x_0, x_1] \cup [x_1, f(x)] = [x_0, f(x)]$. Consequently, $x \in [x_0, f(x))$, i.e. $x < f(x)$.

The union P of the family $\mathscr{P}_0$ is a ray. If there exists $[x_0, x_\alpha) \in \mathscr{P}_0$ which contains all other elements of the family $\mathscr{P}_0$, then $P = [x_0, x_\alpha)$. In the opposite case, let $P = \bigcup_{\alpha \in \mathscr{A}} [x_0, x_\alpha) = \bigcup_{\alpha \in \mathscr{A}} [x_0, x_\alpha]$. Since P is the union of the monotone family of arcs $[x_0, x_\alpha]$, which are closed subsets of X, and this family can be considered as well ordered (taking possibly a cofinal subfamily with the same union P), it follows that it is countable (see [9], vol. I, p. 258, Theorem 3). Thus there exists an increasing sequence of arcs $[x_0, x_{\alpha_n}]$, $n = 1, 2, \ldots$, such that $\bigcup_{n=1}^{\infty} [x_0, x_{\alpha_n}] = P$. Then P is a one-to-one continuous image of the half-line $[0, \infty)$.

Inclusion $L(P) \subset f(L(P))$.

Case 1. If P is contained in an arc, then $\bar{P} = [x_0, y_0]$ for some $y_0 \in X$ and $L(P) = \{y_0\}$. It is to be proved that $y_0 = f(y_0)$.

Suppose, on the contrary, that

$$y_0 \neq f(y_0) \tag{2.5}$$

and take a sequence of points $x_n \in P$ such that $\lim_n x_n = y_0$. By the continuity of f, there exists n_0 such that $[x_{n_0}, y_0] \cap f([x_{n_0}, y_0]) = \emptyset$. Then for each $n \geqslant n_0$, $[x_n, f(x_n)] \multimap [x_n, f(y_0)]$ (namely the smallest arc between $[x_{n_0}, y_0]$ and $f([x_{n_0}, y_0])$ is contained in the common part of the arcs $[x_n, f(x_n)]$ and $[x_n, f(y_0)]$). Simultaneously, by (2.2), $[x_n, y_0] \multimap [x_n, f(x_n)]$, and hence $[x_n, y_0] \multimap [x_n, f(y_0)]$ by the transitivity of the relation $\multimap$. But $[x_0, x_n] \subset [x_0, y_0]$, and thus $[x_0, x_n] \subset [x_0, f(y_0)]$ for all $n \geqslant n_0$ (see [11], p. 108, Proposition 6). Consequently, $[x_0, y_0] = \overline{\bigcup_{n=n_0}^{\infty} [x_0, x_n]} \subset [x_0, f(y_0)]$, i.e. $y_0 \leqslant f(y_0)$. Hence, in view of (2.5), $y_0 < f(y_0)$, i.e. $[x_0, y_0] \subsetneq [x_0, f(y_0)]$.

Now, it is seen that this contradicts the maximality of $P = [x_0, y_0)$ in $\mathscr{P}_0$.

Indeed, as in the proof that the family $\mathscr{P}_0$ is non-empty, there exists an arc $[y_0, y_1] \subset [y_0, f(y_0)]$ such that $y < f(y)$ for all $y \in [y_0, y_1)$. But $[x_0, y_0) \subsetneq [x_0, y_1)$ – a contradiction.

Case 2. If P is not contained in any arc, then, by Lemma 2, $P(x)$ is not contained in any arc for every $x \in P$. Thus, by (2.2), there exists a point $x^* \in P(x) - \{x\}$ such that $P(x) \cap [x, f(x)] = [x, x^*]$. It will be now proved that

$$P(x^*) \subset f(P(x)). \tag{2.6}$$

Take an arbitrary $z \in P(x^*)$. Then

$$z \in [x^*, f(z)] \subset [f(x), f(z)] \tag{2.7}$$

the first relation being trivial, for $z < f(z)$. For the second, observe first that the following inclusion holds:

$$[x^*, z] \subset [f(x), z].$$

This inclusion holds of course in the case where $x^* = f(x)$, and, in the opposite case, $f(x) \notin [x, z]$. Take the smallest arc joining $f(x)$ and $[x, z]$ in X, which is of the form $[f(x), x^*]$. Then $[f(x), x^*] \subset [f(x), z]$, whence $x^* \in [f(x), z]$, i.e. $[x^*, z] \subset [f(x), z]$.

Adding the arc $[z, f(z)]$ to both sides of the inclusion, one gets $[x^*, f(z)] \subset [x^*, z] \cup [z, f(z)] \subset [f(x), z] \cup [z, f(z)]$. But $[f(x), z] \cup [z, f(z)] = [f(x), f(z)]$ because $z \in [x^*, f(z)]$, which proves (2.7).

Since $[f(x), f(z)] \subset f([x, z])$, (2.7) implies (2.6).

The required inclusion for the continuum $L(P) = \bigcap_{n=1}^{\infty} \overline{P(x_n)}$ follows easily, by applying (2.6) to x_n and x_n^* in place of x and x^*:

$$\bigcap_{n=1}^{\infty} \overline{P(x_n^*)} \subset \bigcap_{n=1}^{\infty} \overline{f(P(x_n))}, \tag{2.8}$$

and, by the continuity of f,

$$\bigcap_{n=1}^{\infty} \overline{f(P(x_n))} = \bigcap_{n=1}^{\infty} f(\overline{P(x_n)}) = f(\bigcap_{n=1}^{\infty} \overline{P(x_n)}). \tag{2.9}$$

Finally, if P has small branches, then $f(P(x_n)) \subset Q(x_n)$, where $Q(x_n)$ is the arcwise component of $X - \{x_n\}$ which contains $P(x_n)$. Since $L(P) = \bigcap_{n=1}^{\infty} \overline{Q(x_n)}$, hence, in view of (2.9), $L(P) = f(L(P))$, which completes the proof.

Corollary 2. *Let X be a uniquely arcwise connected continuum and let $x_0 \in X$ define the order $<$ in X.*

(a) *If f: $X \to X$ is continuous and has no fixed point, then there exists a unique ray $P \subset X$ with the initial point x_0 which is not contained in any arc and $x < f(x)$ for all $x \in P$.*

(b) *If every ray $P \subset X$ with the initial point x_0 has no branches and $L(P)$ has the fixed point property, then X has the fixed point property.*

Corollary 3 (cf. [13]). *If X is a uniquely arcwise connected continuum such that every ray of X is contained in an arc, then X has the fixed point property; in particular, every dendroid has the fixed point property.*

Proof. A continuum X is a dendroid if X is arcwise connected and hereditarily unicoherent, i.e. if for every two points $x, y \in X$ there exists a unique continuum irreducible between x and y and this irreducible continuum is an arc $[x, y]$.

Every ray P in X is the union of increasing sequence of arcs: $P = \overline{\bigcup_{n=1}^{\infty} [x_0, x_n]}$. If a proper subcontinuum of the continuum $\overline{\bigcup_{n=1}^{\infty} [x_0, x_n]}$ contains x_0, then there exists x_n which does not belong to this subcontinuum. Hence the continuum $\overline{\bigcup_{n=1}^{\infty} [x_0, x_n]}$ cannot be decomposed into a union of two proper subcontinua which both contain x_0, and thus $\overline{\bigcup_{n=1}^{\infty} [x_0, x_n]}$ is an arc (see [9], vol. II, p. 192, Theorem 4).

Remark 2. (a) Theorem 2 is true, by the same proof, for Hausdorff continua, understanding that a ray is any union of monotone family of arcs with the same end point.

(b) The inclusion in Theorem 2 may be essential as the example constructed in [17] shows.

3. On the class of uniquely arcwise connected curves with the fixed point property. If X and Y are uniquely arcwise connected curves with the fixed point property and $X \cap Y$ has the fixed point property, then $X \cup Y$ need not have the fixed point property (cf. [19], p. 156, Example 3). The necessary condition for $X \cup Y$ to have the fixed point property is of course the arcwise connectedness of $X \cap Y$, because otherwise $X \cup Y$ would contain a simple closed curve. However, this condition is not sufficient – it will be proved in this section that even if $X \cap Y$ has the fixed point property and is arcwise connected, then $X \cup Y$ may not have the fixed point property.

The proof of this fact will be provided by a suitable construction of curves X_1 and Y_1. The curve X_1 will consist of Young's uniquely arcwise connected curve X_0 and a convergent sequence of arcs having only one end point common with X_0.

To describe X_1, and then prove the fixed point property of X_1, recall Young's example [20] introducing necessary notation. Young's curve lies essentially in 3-dimensional Euclidean space E^3, but it will be here described with only one arc lying out of the Euclidean plane E^2. Thus take a cycle $C \subset E^2$ which is the union of two curves homeomorphic to the $\sin\frac{1}{x}$ curve so that the end point of one $\sin\frac{1}{x}$ curve is an end point of the segment of condensation of the other. The segments of condensation place on a straight line and the end points of the segments, which are simultaneously end points of the $\sin\frac{1}{x}$ curves, denoted by a and b. Moreover, let C be symmetric with the center c of the symmetry ψ_c lying on the straight segment $\overline{[a, b]}$. The continuum C described in such a way is the union of rays A and B, which are arc components of C:

$$C = A \cup B \quad \text{and} \quad A \cap B = \emptyset, \tag{3.1}$$

where A and B have initial points a and b respectively. Let $S \subset E^2 - C$ be a spiral approximating C in the unbounded domain of $E^2 - C$ and let s be the initial point of S. Take an arbitrary arc $S_0 \subset E^3$ having as its end points only c and s in E^2 and straight-line segments $A_0 = \overline{[a, c]}$ and $B_0 = \overline{[b, c]}$. Then define

$$X_0 = A_0 \cup B_0 \cup S_0 \cup S \cup C,$$

the Young's uniquely arcwise connected curve (which does not have the fixed point property).

Observe that

$$\bar{A} - A \text{ is an arc in } B \text{ and } \bar{B} - B \text{ is an arc in } A \tag{3.2}$$

and take a point $a^* \in A - (\bar{B} - B)$. The point a^* separates A into two connected subsets: the arc $[a, a^*]$ and the ray A^* with the initial point a^*:

$$\overline{A - A^*} = [a, a^*] \subset A \quad \text{and} \quad A^* \subset A. \tag{3.3}$$

The continuum X_1 will be now defined as follows:

$$X_1 = X_0 \cup \bigcup_{n=1}^{\infty} A_n$$

where $A_n = [b_n, c_n] \subset E^2$ and $A_n \cap X_0 = \{b_n\}$ for each $n = 1, 2, \ldots$ and

$$b_n \in B_0 \quad \text{and} \quad \lim_n A_n = \overline{A^*}. \tag{3.4}$$

Such a sequence of arcs A_n can be easily constructed taking into acount that the bounded domain of $E^2 - C$ is the union of a strongly increasing sequence of disks so that the boundaries of these disks lead to the desired arcs A_n.

Though the closure in (3.2)–(3.4) was considered with respect to plane topology, it concerns the subsets of the continuum X_1, and thus it may be further considered as the closure in X_1.

THEOREM 3. *The curve X_1 has the fixed point property.*

Proof. Suppose, on the contrary, that there exists a continuous transformation $f\colon X_1 \to X_1$ without any fixed point. Take $c \in X_1$ as the initial point for the order in X_1, so that every ray in X_1 has not branches. Then by Theorem 2 there exists a ray $P \subset X_1$ such that $L(P) = f(L(P))$. Since $S_0 \cup S = P$ is the only ray with the continuum $L(P)$ without the fixed point property, and $L(P) = C$, hence, in view of (3.1), $A \cup B = f(A \cup B)$. It follows, in view of Theorem 1, that $f(B) = A$ and

$$f(A) = B. \tag{3.5}$$

Moreover

$$f(\bar{A} - A) = \bar{B} - B. \tag{3.6}$$

Indeed, $f(\bar{A} - A) \cap B = \emptyset$ by (3.1) and (3.2), and $f(\bar{A} - A) \subset \bar{B}$ by the continuity of f. Hence $f(\bar{A} - A) \subset \bar{B} - B$. The converse inclusion follows from [5] ((3), p. 28).

Denote now for every $n = 1, 2, \ldots$

$$C_n = \bigcup_{k=n}^{\infty} A_k \cup [b_k, b_{k+1}],$$

so that every C_n is an arcwise connected set and, by (3.4),

$$\lim C_n = \overline{A^*}. \tag{3.7}$$

It will be now proved that

$$f(C_n) \subset X - \{c\} \quad \text{for some } n. \tag{3.8}$$

If, on the contrary, $c \in f(A_{n_k} \cup [b_{n_k}, b_{n_k+1}])$ for some sequence $n_1 < n_2 < \ldots$, i.e. $c = f(x_{n_k})$ where $x_{n_k} \in A_{n_k} \cup [b_{n_k}, b_{n_k+1}]$ for $k = 1, 2, \ldots$, then for a convergent subsequence $\lim x_{n_{k_i}} \in \bar{A}$ by (3.3) and (3.4). Thus $c \in f(\bar{A})$ contradicting (3.5) and (3.6), which proves (3.8).

It follows from (3.8), in view of (3.2), (3.5) and (3.6), that $f(C_n \cup (\bar{A} - A))$ is contained in the same arc component of the set $X - \{c\}$. But $\bar{B} - B \subset f(C_n \cup (\bar{A} - A))$ by (3.6) and $\bar{B} - B \subset A$ by (3.2). Consequently

$$f(C_n) \subset A \cup A_0,$$

and hence $f(\bar{C}_n) \subset A_0 \cup A \cup (\bar{A} - A)$ by (3.3) and (3.7). Thus $f(A^*) \subset \bar{A} - A$ in view of (3.1), (3.3), (3.5) and (3.7). Since $\bar{A} - A$ is an arc in B by (3.2) and also $f(\overline{A - A^*})$ is a compact subset of B by (3.2), (3.3) and (3.5), it follows that $f(A)$ is contained in some arc of B, contradicting (3.5). This proves Theorem 3.

Since the continua

$$Y_1 = A_0 \cup B_0 \cup \bar{B} \cup \psi_c\Big(\bigcup_{n=1}^{\infty} A_n\Big)$$

and $X_1 \cap Y_1 = A_0 \cup B_0 \cup \bar{B}$ have the fixed point property in view of Corollary 2(b), and, similarly to the Young's continuum X_0, $X_1 \cup Y_1$ does not have the fixed point property, we have the following

COROLLARY 4. *There exist uniquely arcwise connected curves X, Y and*

$X \cap Y$ which have the fixed point property, but the curve $X \cup Y$ does not have the fixed point property.

The following problem arises to generalize [18]:

PROBLEM 2. Suppose X and Y are uniquely arcwise connected curves with the fixed point property and $X \cap Y$ is a dendroid. Must $X \cup Y$ have the fixed point property? **(P 1297)**

Finally, observe how Theorem 2 can be used to prove the fixed point property of Bing's uniquely arcwise connected curve X_2, which can be defined in our denotation as follows. Let $S_0^* = [a^*, s] \subset E^3$ be an arbitrary arc having only the end points on the plane E^2. Then $X_2 = A_0 \cup B_0 \cup S_0^* \cup S \cup C$.

Applying Theorem 2 to $X = X_2$ and $x_0 = a^*$, the inequality $a^* \neq f(a^*)$ implies $a^* < f(a^*)$ and $f(a^*) \in C$ which is impossible. Thus we obtain

COROLLARY 5 (cf. [2], p. 124, Theorem 14). *Bing's uniquely arcwise connected curve X_2 has the fixed point property.*

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