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# ON SPACES WITH BINARY NORMAL SUBBASE

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A collection of closed subsets $\mathcal{S}$ of a topological space $X$ is called a *closed subbase* provided that for each closed set $A \subset X$ and for each point $x \notin A$ there is a finite $F \subset \mathcal{S}$ such that $A \subset \bigcup F$ and $x \notin \bigcup F$.

A closed subbase $\mathcal{S}$ for a topological space is called *binary* provided that for all $L \subset \mathcal{S}$ with $\bigcap L = \emptyset$ there are $L_1, L_2 \in L$ with $L_1 \cap L_2 = \emptyset$. In addition, the subbase $\mathcal{S}$ is called *normal* if for all $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \emptyset$ there are $S_1', S_2' \in \mathcal{S}$ with $S_1 \subset S_1' - S_2'$, $S_2 \subset S_2' - S_1'$, and $S_1' \cup S_2' = X$.

For the beautiful treatment of spaces which admit a binary subbase we refer to [4].

In this note we consider spaces admitting a subbase which is simultaneously binary and normal, i.e., a *binary normal subbase*. The products of compact orderable spaces and the products of compact tree-like spaces admit a binary normal subbase [2]. In the case of connected metric spaces with a binary normal subbase van Mill [3] proved that they are AR's and he raised there a question of whether every compact metric AR admits a binary normal subbase. In the case of 1-dimensional compact metric AR's this question has a positive answer [3]. We answer this question in the negative for higher dimensions by showing that the Borsuk 2-dimensional compact metric AR having the singularity of Mazurkiewicz ([1], p. 152) is a counterexample.

Since for each pair of distinct points $a, b$ of a $T_1$-space $X$ with a binary subbase $\mathcal{S}$ there exist $A, B \in \mathcal{S}$ with $a \in A$, $b \in B$, and $A \cap B = \emptyset$, we have

LEMMA. *If $\mathcal{S}$ is a binary normal subbase in a $T_1$-space $X$, then for each pair $a, b$ of distinct points of $X$ there exist $A, B \in \mathcal{S}$ with $a \in A - B$, $b \in B - A$, and $A \cup B = X$.*

THEOREM. *Let $\mathcal{S}$ be a binary normal subbase in a $T_1$-space $X$. If $R \subset \mathcal{S}$ is such that $\bigcap R \neq \emptyset$, then $\bigcap R$ is a retract of $X$.*

Proof. For $x \in X$ let $L(x, R)$ be the family of all members of $\mathcal{S}$ which contain $x$ and intersect $\bigcap R$. Clearly, the family $R \cup L(x, R)$ is contained in $\mathcal{S}$ and every two members of that family meet. Since $\mathcal{S}$ is binary,

$\bigcap R \cap \bigcap L(x, R)$ is non-empty. Moreover, $\bigcap R \cap \bigcap L(x, R)$ is a singleton from $\bigcap R$. In fact, take arbitrary different points $a, b \in \bigcap R$. By the Lemma, there exist $A, B \in \mathscr{S}$ with $a \in A - B$, $b \in B - A$, and $A \cup B = X$. One of $A$ and $B$, say $A$, contains $x$. Since $a$ belongs to $\bigcap R$, $A \in L(x, R)$. Hence $b \notin \bigcap R \cap \bigcap L(x, R)$.

Define a map $r: X \to \bigcap R$ by $r(x) = \bigcap R \cap \bigcap L(x, R)$. Clearly, $r(x) = x$ for each $x \in \bigcap R$. It remains to prove the continuity of $r$.

Let $S \in \mathscr{S}$ be such that $\bigcap R \cap S \neq \emptyset$. Then $S \in L(x, R)$ for each $x \in S$. Hence $\bigcap R \cap \bigcap L(x, R) \subset \bigcap R \cap S$ for each $x \in S$. In other words,

(*) $r(S) \subset \bigcap R \cap S$ whenever $S \in \mathscr{S}$ and $\bigcap R \cap S \neq \emptyset$.

Now, we shall prove that $r^{-1}(\bigcap R \cap S)$ is closed in $X$ for each $S \in \mathscr{S}$ such that $\bigcap R \cap S \neq \emptyset$.

Let $x \notin r^{-1}(\bigcap R \cap S)$. Then $\bigcap R \cap \bigcap L(x, R) \cap S = \emptyset$. Since $\mathscr{S}$ is binary, there exist $A, B \in R \cup L(x, R) \cup \{S\}$ with $A \cap B = \emptyset$. This is possible only in the case where $A \in L(x, R)$ and $B = S$. Since $\mathscr{S}$ is normal, there exist $A', B' \in \mathscr{S}$ such that $A \subset A' - B'$, $B \subset B' - A'$, and $A' \cup B' = X$. Hence $X - B'$ is an open neighborhood of $x$ contained in $A'$. This neighborhood of $x$ is disjoint with $r^{-1}(\bigcap R \cap S)$.

Indeed, since $A' \cap B = \emptyset$ and $B = S$, we have

$$r^{-1}(\bigcap R \cap A') \cap r^{-1}(\bigcap R \cap S) = \emptyset.$$

Since $A \subset A'$, $\bigcap R \cap A' \neq \emptyset$. Hence, by (*),

$$A' \subset r^{-1}(r(A')) \subset r^{-1}(\bigcap R \cap A');$$

in consequence, $X - B' \subset A' \subset r^{-1}(\bigcap R \cap A')$. Hence

$$(X - B') \cap r^{-1}(\bigcap R \cap S) = \emptyset.$$

Since $\mathscr{S}$ is a closed subbase in $X$, $r$ is continuous.

It follows from the Lemma that each open neighborhood of an arbitrary point from a $T_1$-space $X$ with a binary normal subbase $\mathscr{S}$ contains the intersection of finitely many neighborhoods of $x$ belonging to $\mathscr{S}$. By the Theorem, that intersection is a retract of $X$. In the case where $X$ is connected, that intersection is also connected. Thus we get

COROLLARY 1 (Verbeek [5]). *If a $T_1$ connected space has a binary normal subbase, then it is locally connected.*

The same arguments in the case where $X$ is an AR lead to

COROLLARY 2. *If $X$ is an AR and has a binary normal subbase, then each neighborhood of an arbitrary point of $X$ contains a neighborhood of that point which is an AR.*

Corollary 2 for metric AR's states that metric AR's with a binary normal subbase do not have, in particular, the singularity of Mazurkiewicz. Recall (following Borsuk [1], p. 152) that a metric space which cannot be expressed as a finite or countable union of AR-sets of arbitrarily small diameter is said to have the *singularity of Mazurkiewicz*. Borsuk [1] constructed a 2-dimensional compact metric AR with this singularity. In consequence, this AR does not have a binary normal subbase.

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