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COMPACT KAEHLER MANIFOLDS AND THE EIGENVALUES OF THE LAPLACIAN

BY

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0. Introduction. Let (M, g) be an n-dimensional compact Kaehler manifold and let Δ be the Laplacian of (M, g). Using a local complex coordinate system $(x_1, \ldots, x_n)$, we express Δ as

$$\Delta = -2 \sum_{i,j=1}^{n} g^{\bar{i}j}(\partial/\partial x_{\bar{i}})(\partial/\partial x_j), \tag{0.1}$$

where $(g^{\bar{i}j})$ is the inverse matrix of $g = (g_{\bar{i}j})$. We denote by $\lambda_1(M)$ the first eigenvalue of Δ of M. Then the following result is known:

THEOREM A ([6]). *Let M be an n-dimensional compact Einstein Kaehler manifold of positive scalar curvature τ. Then*

$$\lambda_1(M) \geqslant \tau/n.$$

The equality holds if and only if M admits a one-parameter group of isometries (i.e., a non-trivial Killing vector field).

Theorem A can be proved by using a Bochner type identity. Then, using the same method, we can show the following

THEOREM 1. *Let M be an n-dimensional compact Kaehler manifold Assume that the Ricci curvature* Ricci *of M satisfies*

$$\mathrm{Ricci} \geqslant k \ (> 0).$$

Then

$$\lambda_1(M) \geqslant 2k. \tag{0.2}$$

If the equality holds, then M admits a non-trivial Killing vector field.

Remark 1. Urakawa [10] also proved Theorem 1 by using a different method.

In Theorem 1, if $\lambda_1(M) = 2k$, then the Ricci tensor Ric of M satisfies

(0.3) $\quad \mathrm{Ric}(x, x) = kg(x, x)$ for some non-trivial vector field X.

We put $\mathrm{Ric}_0 = \mathrm{Ric} - kg$. Then we consider the nullity distribution

$$D = \{X \in T_p M \mid \mathrm{Ric}_0(X, Y) = 0 \text{ for all } Y \in T_p M\}$$

at $p \in M$. The set M_0 where $\dim D$ attains its minimum is an open subset of M. The problem is whether the subbundle of $D|_{M_0}$ of the tangent bundle is integrable or not. We do not know whether it is integrable or not. However, Ferus' general result [2] states that the condition for $D|_{M_0}$ to be integrable is that the Ricci tensor is of Codazzi type, which implies that M has the parallel Ricci tensor since M is a Kaehler manifold. On the other hand, we have the following

THEOREM 2. *Let M be an n-dimensional compact simply-connected Kaehler manifold with parallel Ricci tensor. Assume that there is a non-trivial Killing vector field on M. Then*

$$\lambda_1(M) \leqslant (n+1)H, \tag{0.4}$$

where H is the maximum of holomorphic sectional curvature of M. Moreover, the equality holds if and only if M is complex-analytically isometric to a complex projective space of constant holomorphic sectional curvature H.

Remark 2. By Kobayashi's Theorem [4], a compact Kaehler manifold of positive Ricci tensor is simply connected. In Theorem 2, if M is irreducible, then $\lambda_1(M) = \tau/n$ (> 0).

Next, we investigate the sufficient conditions for M to be a complex projective space. Recently, Kameda and Yamaguchi [3] showed the following

THEOREM B. *Let M be an n-dimensional compact Kaehler manifold. Assume that there is a non-constant function f such that $\Delta f = \lambda f$ for some real constant λ and that f has the property*

$$2\,\mathrm{Ric}(\mathrm{grad}(f), \mathrm{grad}(f)) \geqslant \lambda g(\mathrm{grad}(f), \mathrm{grad}(f)),$$

where $\mathrm{grad}(f)$ *is the gradient of f. Then*

$$\lambda \geqslant 4(n+1)k_0, \tag{0.5}$$

where k_0 is the minimum of sectional curvature of M. Moreover, the equality holds if and only if M is complex-analytically isometric to a complex projective space of constant holomorphic sectional curvature $4k_0$.

Remark 3. It is proved by Siu and Yau [7] that every compact Kaehler manifold of positive bisectional curvature is biholomorphic to a complex projective space.

With reference to Theorem B, we have the following

THEOREM 3. *Let M be an n-dimensional compact Kaehler manifold. Assume that there is a non-constant function f such that $\Delta f = \lambda f$ for some real* *constant λ and that f has the properties*

(i) $$\int_M \left(2\operatorname{Ric}(\operatorname{grad}(f), \operatorname{grad}(f)) - \lambda g(\operatorname{grad}(f), \operatorname{grad}(f))\right) \geqslant 0,$$

(ii) $$\int_M \left(2\operatorname{tr}(\operatorname{Ric}\cdot\operatorname{Hess}(f)\cdot\operatorname{Hess}(f)) - \lambda |\operatorname{Hess}(f)|^2\right) \geqslant 0,$$

where $\operatorname{Hess}(f)$ *is the Hessian of f, and $\cdot$ is the matrix product of* $\operatorname{Ric} = (R_{i\bar{j}})$, $\operatorname{Hess}(f) = (f_{AB})$ *for $i, j = 1, \ldots, n$, and $A, B = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$. Then*

(0.6) $$\lambda \leqslant (n+1)K,$$

where K is the maximum of holomorphic bisectional curvature of M. Moreover, the equality holds if and only if M is complex-analytically isometric to a complex projective space of constant holomorphic sectional curvature K.

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1. Preliminaries. We use the following convention on the range of indices:

$$i, j, k, \ldots = 1, \ldots, n, \qquad A, B, C, \ldots = 1, \ldots, n, \bar{1}, \ldots, \bar{n}.$$

We choose a local field of unitary frames $e_1, \ldots, e_n$ on M and denote by $f_A, f_{AB}, f_{ABC}, \ldots$ the first ordered covariant derivative with respect to e_A, the second ordered covariant derivative with respect to e_A and e_B, the third ordered covariant derivative with respect to e_A, e_B and e_C of a real function f, respectively. Let $R = (R_{ABCD})$ be the curvature tensor of M. The following relations are well known:

(1.1) $$f_{AB} = f_{BA}$$

(i.e., a Hessian matrix is symmetric),

(1.2) $$R_{ABij} = R_{AB\bar{i}\bar{j}} = R_{ijAB} = R_{\bar{i}\bar{j}AB} = 0, \qquad R_{\bar{i}jk\bar{l}} = R_{\bar{l}jk\bar{i}} = R_{\bar{i}kj\bar{l}},$$

(1.3) $$f_{ABC} - f_{ACB} = -\sum_D R_{DABC} f_{\bar{D}} \quad \text{(Ricci identity)}.$$

Then we can easily see that

(1.4) $f_{A_1\ldots A_m ij}$ and $f_{A_1\ldots A_m \bar{i}\bar{j}}$ are symmetric with respect to i and j for any $A_1, \ldots, A_m$.

The Ricci tensor $\operatorname{Ric} = (R_{i\bar{j}})$ and the scalar curvature τ are defined by

$$R_{i\bar{j}} = \sum_k R_{\bar{j}k\bar{k}i} \quad \text{and} \quad \tau = 2\sum_i R_{i\bar{i}}.$$

2. An integral formula on the unit tangent bundle. Let TM^C be the complexification of the tangent bundle of M. Then we have $TM^C = TM^+$

$+TM^-$ (orthogonal sum), where TM^+ (resp., TM^-) is the $(\sqrt{-1})$-eigenspace (resp., $(-\sqrt{-1})$-eigenspace) of the complex structure tensor J of M. We denote by U_pM^+ the unit sphere of T_pM^+ for $p \in M$. The following lemma is essentially due to Willmore [12].

LEMMA 1. *Let T be any C-valued (k, k)-type tensor, i.e.,*

$$T = (T_{i_1 \dots i_k j_1 \dots j_k}),$$

and

$$T_{i_1 \dots i_k j_1 \dots j_k} = T(e_{i_1}, \dots, e_{i_k}; e_{\bar{j}_1}, \dots, e_{\bar{j}_k}).$$

Then

$$\int_{U_pM^+} T(u, \dots, u; \bar{u}, \dots, \bar{u}) = L \sum_{\sigma} \sum_{i_1, \dots, i_k = 1}^{n} T(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}; e_{\bar{i}_1}, \dots, e_{\bar{i}_k}), \tag{2.1}$$

where $L = (n(n+1)\dots(n+k-1))^{-1} \operatorname{vol}(U_pM^+)$ and the summation on σ is taken over all permutations of $(1, \dots, k)$.

In fact, by SU(n)-invariant theory [8], the left-hand side of (2.1) is generated by Hermitian inner products. The coefficient of the right-hand side of (2.1) is determined by using Weyl's tube formula [11]. As an easy application of Lemma 1, we have the following

PROPOSITION 1. *Let M be an n-dimensional compact Kaehler manifold. Then*

$$\lambda_1(M) \geqslant (n+1)(n+2)H_0/2 - \tau_{\max}/2,$$

where H_0 is the minimum of holomorphic sectional curvature and $\tau_{\max}$ is the maximum of the scalar curvature of M.

We prove Proposition 1 in Section 3.

3. Proofs of the results. From Green's Theorem we obtain

$$0 = \int_M \sum_{i,j} (f_{ij} f_{\bar{i}})_{\bar{j}} = \int_M \sum_{i,j} (|f_{ij}|^2 + f_{ij\bar{j}} f_{\bar{i}}). \tag{3.1}$$

Then, using (1.1)–(1.3), we have

$$\sum_j f_{ij\bar{j}} = \sum_j f_{ji\bar{j}} = \sum_j f_{j\bar{j}i} - \sum_{j,l} R_{\bar{l}ij\bar{j}} f_l = (-1/2)(\Delta f)_i + \sum_l R_{\bar{l}i} f_l. \tag{3.2}$$

Therefore, if $\Delta f = \lambda f$, (3.1) and (3.2) imply

$$\int_M \Big(\sum_{i,j} |f_{ij}|^2 - (\lambda/2) \sum_i |f_i|^2 + \sum_{i,l} R_{\bar{l}i} f_l f_{\bar{i}}\Big) = 0. \tag{3.3}$$

If M is Einstein, then $R_{i\bar{i}} = (\tau/2n)\delta_{ii}$, which, together with (3.3), implies

$$0 \leqslant \int_M \sum_{i,j} |f_{ij}|^2 = ((n\lambda - \tau)/2n) \int_M \sum_i |f_i|^2. \tag{3.4}$$

This yields

$$\lambda_1(M) \geqslant \tau/n. \tag{3.5}$$

Moreover, if (Ricci curvature) $\geqslant k$ (> 0), then

$$\sum_{i,l} R_{i\bar{l}} f_l f_{\bar{i}} \geqslant k \sum_i |f_i|^2,$$

which, together with (3.3), implies

$$0 \leqslant \int_M \sum_{i,j} |f_{ij}|^2 \leqslant ((\lambda - 2k)/2) \int_M \sum_i |f_i|^2. \tag{3.6}$$

This yields

$$\lambda_1(M) \geqslant 2k. \tag{3.7}$$

LEMMA 2. *Assume that $B_1(M) = 0$, where $B_1(M)$ denotes the first Betti number of M. Then there is a non-constant function f such that $f_{ij} = 0$ for any i and j if and only if there is a non-trivial Killing vector field on M.*

Lemma 2, together with (3.4)–(3.7) and Remark 2, implies Theorem A and Theorem 1.

Proof of Lemma 2. We can define the Laplacian acting on 1-forms as follows (see [5]):

$$\Delta(f_i) = -\sum_j (f_{ij\bar{j}} + f_{i\bar{j}j}) + \sum_j R_{i\bar{j}} f_j. \tag{3.8}$$

Then, using (1.1), (1.3) and (1.4), we obtain

$$\Delta(f_i) = 2\sum_j R_{i\bar{j}} f_j - 2\sum_j f_{ij\bar{j}}. \tag{3.9}$$

Since M is compact, it follows from (3.1) and (3.9) that

$$(f_{ij} = 0 \text{ for any } i \text{ and } j) \Leftrightarrow \Big(\sum_j f_{ij\bar{j}} = 0 \text{ for any } i\Big)$$
$$\Leftrightarrow \Delta(f_i) = 2\sum_j R_{i\bar{j}} f_j \text{ for any } i.$$

Then the following result of Matsushima [5] implies Lemma 2.

THEOREM C. *Let M be a compact Kaehler manifold such that $B_1(M) = 0$. Then all the Killing forms on M can be represented as $J\,\mathrm{grad}(f)$, where J is* *the complex structure tensor of M and f satisfies*

$$\Delta(f_i) = 2\sum_j R_{i\bar{j}} f_j \quad \textit{for any } i.$$

Next, we prove Proposition 1.

For any unit vector u of $U_p M^+$, we consider the following function $F(u)$ on $U_p M^+$:

$$F(u) = R_{\bar{u}u\bar{u}u} f_u f_{\bar{u}},$$

where $R_{\bar{u}u\bar{u}u}$ is the holomorphic sectional curvature of the plane spanned by u and $\bar{u}$, and f_u is the gradient of f with respect to u. Then, from (2.1) we obtain at $p \in M$

$$\text{(3.10)} \quad \int_{U_pM^+} F(u) = \big(n(n+1)(n+2)\big)^{-1} \operatorname{vol}(U_p M^+)\big(\tau \sum_i |f_i|^2 + 4 \sum_{i,j} R_{i\bar{j}} f_{\bar{i}} f_j\big).$$

On the other hand, we have

$$\text{(3.11)} \quad \int_{U_pM^+} F(u) \geqslant H_0 \int_{U_pM^+} f_u f_{\bar{u}} = \big\{(H_0 \operatorname{vol}(U_p M^+))/n\big\} \sum_i |f_i|^2.$$

Therefore, we get

$$\text{(3.12)} \quad (n+1)(n+2) H_0 \int_M \sum_i |f_i|^2 \leqslant \int_M \big(\tau \sum_i |f_i|^2 + 4 \sum_{i,j} R_{i\bar{j}} f_{\bar{i}} f_j\big)$$
$$\leqslant \tau_{\max} \int_M \sum_i |f_i|^2 + 4 \int_M \sum_{i,j} R_{i\bar{j}} f_{\bar{i}} f_j.$$

This, together with (3.3), yields the desired assertion.

Next, we prove Theorem 2.

First, from Theorem C we see that there is a non-constant function f such that

$$\text{(3.13)} \quad (\Delta f)_k = 2 \sum_i R_{\bar{i}k} f_i$$

because of (0.1). Let $M = M_1 \times \ldots \times M_s$ be the de Rham decomposition of M. Then each factor M_α $(\alpha = 1, \ldots, s)$ is a compact simply-connected Einstein Kaehler manifold. Let

$$f = f^1 \times \ldots \times f^s$$

be the corresponding decomposition of f. We denote by n_α and τ_α the dimension and the scalar curvature of M_α, respectively. Then it follows from (3.13) that

$$(\Delta f^\alpha)_k = (\tau_\alpha / n_\alpha) f_k^\alpha \quad \text{for each } \alpha.$$

Since

$$(1/2)\int_{M_\alpha}|\Delta f^\alpha|^2=\int_{M_\alpha}\sum_k(\Delta f^\alpha)_k f^\alpha_{\bar k}=(\tau_\alpha/n_\alpha)\int_{M_\alpha}\sum_k|f^\alpha_k|^2,$$

we see that if $\tau_\alpha\leqslant 0$, then $f^\alpha=\text{const}$ on M_α. Therefore, from the non-constancy of f it follows that there exists $\beta\in(1,\ldots,s)$ such that f^β is not constant and $\tau_\beta>0$ and there is a non-trivial Killing vector field on M_β. This implies that

$$\lambda_1(M_\beta)=\tau_\beta/n_\beta$$

(by Theorem A). Note that

$$\tau_\beta/n_\beta(n_\beta+1)\leqslant H_\beta$$

(by Lemma 1), where H_β is the maximum of the holomorphic sectional curvature of M_β. Then

$$\lambda_1(M)\leqslant\lambda_1(M_\beta)=\tau_\beta/n_\beta\leqslant(n_\beta+1)H_\beta\leqslant(n+1)H.$$

If the equality holds, then $M=M_\beta$ (i.e., M is irreducible) and M has the constant holomorphic sectional curvature $H=H_\beta$. The converse is well known. Then Theorem 2 is proved.

Finally, we prove Theorem 3.

The assumption (i), together with (3.3), implies

$$f_{ij}=0\quad\text{for any } i \text{ and } j. \tag{3.14}$$

From Green's Theorem we have

$$0=\int_M\sum_{i,j,k}(f_{i\bar j k}f_{\bar i j})_{\bar k}=\int_M\sum_{i,j,k}(|f_{i\bar j k}|^2+f_{i\bar j k\bar k}f_{\bar i j}). \tag{3.15}$$

On the other hand, it follows from (1.3) and (3.14) that

$$\sum_k f_{i\bar j k\bar k}=\sum_k(f_{ik\bar j}-\sum_l R_{\bar l i\bar j k}f_l)_{\bar k}=-\sum_l R_{\bar l i;\bar j}f_l-\sum_{l,k}R_{\bar l i\bar j k}f_{l\bar k} \tag{3.16}$$

because the second Bianchi identity states that

$$\sum_k R_{\bar l i\bar j k;\bar k}=-\sum_k(R_{\bar l i k\bar k;\bar j}+R_{\bar l i\bar k\bar j;k})=R_{\bar l i;\bar j}.$$

Since

$$(-1/2)\lambda f_i=(-1/2)(\Delta f)_i=\sum_k f_{k\bar k i}=-\sum_l R_{\bar l i}f_l,$$

we obtain

$$\lambda f_{i\bar j}=2\sum_l(R_{\bar l i;\bar j}f_l+R_{\bar l i}f_{l\bar j}). \tag{3.17}$$

This, together with (3.16), implies

$$\sum_k f_{i\bar{j}k\bar{k}} = (-\lambda/2) f_{i\bar{j}} + \sum_l R_{\bar{l}i} f_{l\bar{j}} - \sum_{l,k} R_{\bar{l}i\bar{j}k} f_{l\bar{k}}. \tag{3.18}$$

Substituting (3.18) into (3.15), we obtain

$$\begin{aligned}\int_M \sum_{i,j,k} |f_{i\bar{j}k}|^2 &= \int_M \Big(\sum_{i,j} (\lambda/2) |f_{i\bar{j}}|^2 - \sum_{i,j,l} R_{\bar{l}i} f_{l\bar{j}} f_{\bar{i}j}\Big) \\ &\quad + \int_M \sum_{i,j,k,l} R_{\bar{l}i\bar{j}k} f_{l\bar{k}} f_{\bar{i}j}.\end{aligned} \tag{3.19}$$

Since $(f_{l\bar{k}})$ is a Hermitian matrix, by choosing a suitable unitary frame we can assume that $(f_{l\bar{k}})$ is a diagonal matrix. Then we see that

$$\begin{aligned}\int_M \sum_{i,j,k,l} R_{\bar{l}i\bar{j}k} f_{l\bar{k}} f_{\bar{i}j} &= \int_M \sum_{i,l} R_{\bar{l}i\bar{i}l} f_{l\bar{l}} f_{i\bar{i}} \\ &\leqslant K \int_M \sum_{i,l} f_{l\bar{l}} f_{i\bar{i}} = (K\lambda/2) \int_M \sum_i |f_i|^2,\end{aligned} \tag{3.20}$$

where K denotes the maximum of holomorphic bisectional curvature of M. This, together with the assumptions (ii) and (3.19), yields

$$\int_M \sum_{i,j,k} |f_{i\bar{j}k}|^2 \leqslant (K\lambda/2) \int_M \sum_i |f_i|^2. \tag{3.21}$$

Following the idea of [1], we put

$$B(f)_{i\bar{j}k} = f_{i\bar{j}k} + (\lambda/2(n+1))(f_i \delta_{jk} + f_k \delta_{ij}). \tag{3.22}$$

Then we can rewrite (3.21) as follows:

$$\int_M \sum_{i,j,k} |B(f)_{i\bar{j}k}|^2 \leqslant (\lambda/2(n+1))((n+1)K - \lambda) \int_M \sum_i |f_i|^2,$$

which yields

$$\lambda \leqslant (n+1)K.$$

If the equality holds, then $B(f)_{i\bar{j}k} = 0$ on M, which implies that

$$f_{i\bar{j}k} + (K/2)(f_i \delta_{jk} + f_k \delta_{ij}) = 0 \quad \text{on } M. \tag{3.23}$$

Then the following result of Obata–Tanno [9] implies that M is complex-analytically isometric to a complex projective space of constant holomorphic sectional curvature K. The converse is trivial.

THEOREM D. *Let M be an n-dimensional complete Kaehler manifold.*

Then there is a non-constant function f on M that satisfies

$$2f_{ijk}+c(f_i\delta_{jk}+f_k\delta_{ij})=0, \quad c>0,$$
$$f_{ijA}=0 \quad \textit{for any } i, j, k \textit{ and } A$$

if and only if M is complex-analytically isometric to a complex projective space of constant holomorphic sectional curvature c.

REFERENCES

[1] K. Benko, M. Kothe, K.-D. Semmler and U. Simon, *Eigenvalues of the Laplacian and curvature*, Colloq. Math. 42 (1979), pp. 19–31.

[2] D. Ferus, *On the completeness of nullity foliations*, Michigan Math. J. 18 (1971), pp. 61–64.

[3] T. Kameda and S. Yamaguchi, *Certain sufficient conditions to be a complex projective space*, Colloq. Math. 49 (1985), pp. 235–241.

[4] S. Kobayashi, *On compact Kaehler manifolds with positive Ricci tensor*, Ann. of Math. 74 (1961), pp. 570–574.

[5] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété Kaehlerienne*, Nagoya Math. J. 11 (1957), pp. 145–150.

[6] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. U.S.-Japan Sem. in Differential Geom., Kyoto, Japan, 1965, pp. 101–114.

[7] Y. T. Siu and S. T. Yau, *Compact Kaehler manifold of positive bisectional curvature*, Invent. Math. 59 (1980), pp. 189–204.

[8] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Inc.

[9] S. Tanno, *Some differential equations on Riemannian manifolds*, J. Math. Soc. Japan 30 (1978), pp. 509–531.

[10] H. Urakawa, *Stability of harmonic maps and eigenvalues of the Laplacian*, Trans. Amer. Math. Soc. 301 (1987), pp. 557–589.

[11] H. Weyl, *On the volume of tubes*, Amer. J. Math. 61 (1939), pp. 461–472.

[12] T. J. Willmore, *The Euclidean Laplacian*, E. B. Christoffel, edited by P. L. Butzer and F. Fehér, Birkhäuser Verlag, Basel 1981, pp. 508–516.

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