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SHAPE EVALUATION GROUPS

BY

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The evaluation subgroups of homotopy groups were introduced and studied by Gottlieb [4]–[7]. In the case of CW-complexes the evaluation subgroups are homotopy invariants and for them there exist several interesting geometric applications (see [1] and [4]–[6]).

In this note we consider a class of pointed topological spaces, later referred to as *D-spaces*, including the pointed topological spaces which have the homotopy type of a pointed CW-complex, and we define their *evaluation pro-groups* and *shape evaluation groups*. These pro-groups and groups are shape invariants. As an application we study the existence of an ε-cross-section for an approximate fibration over the n-sphere.

1. Inverse *D*-systems and *D*-objects. The notions and results on inverse systems and on the shape theory used by us are taken from the book of Mardešić and Segal [9].

Let $\mathscr{P}$ be an arbitrary category and let $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathscr{P}$.

1.1. Definition. We say that the inverse system X is a *D-system* if all bounding morphisms $p_{\lambda\lambda'}$ are domination morphisms, i.e., for each $\lambda, \lambda' \in \Lambda$ with $\lambda \leqslant \lambda'$ there exists a morphism

$$p^*_{\lambda\lambda'}\colon X_\lambda \to X_{\lambda'}$$

such that $p_{\lambda\lambda'} p^*_{\lambda\lambda'} = 1_{X_\lambda}$.

1.2. Example. The sequence $G = (G_n, G_{nn+1})$, with $G_1 = Z$, $G_{n+1} = G_n \oplus Z$ and p_{nn+1} being the first projection, is a D-system of groups.

1.3. Proposition. *Let X be a D-system in a category $\mathscr{P}$. Then X is movable. If $\mathscr{P}$ is a subcategory of the category* Ens_*, *then X has also the Mittag-Leffler property.*

Let $\mathscr{T}$ be an arbitrary category and let $\mathscr{P} \subseteq \mathscr{T}$ be a dense subcategory ([9], p. 25). Denote by $\mathrm{Sh}_{(\mathscr{T},\mathscr{P})}$ the shape category for $(\mathscr{T}, \mathscr{P})$.

1.4. Definition. An object $X \in \mathrm{Sh}_{(\mathscr{T},\mathscr{P})}$ is called a *D-object* if it admits a

$\mathscr{P}$-expansion

$$p\colon X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$$

such that $\boldsymbol{X}$ is a D-system.

1.5. THEOREM. *The D-object property is a shape invariant.*

Proof. Let $X \in \mathrm{Sh}_{(\mathscr{T}, \mathscr{P})}$ be a D-object with a $\mathscr{P}$-expansion $p\colon X \to \boldsymbol{X}$ which is a D-system. If $\mathrm{sh}(Y) = \mathrm{sh}(X)$, there exists a $\mathscr{P}$-expansion $q\colon Y \to \boldsymbol{X}$ (see [10]), and this proves that Y is a D-object.

Generally, the isomorphisms in the category pro-$\mathscr{P}$ do not preserve the D-property of inverse systems.

2. *D*-spaces. Let HTop_* be the pointed homotopy category of pointed topological spaces and let HPol_* be the pointed homotopy full subcategory of polyhedra. The subcategory HPol_* is dense in the category HTop_*, and the shape category for $(\mathrm{HTop}_*, \mathrm{HPol}_*)$ is denoted by Sh_*.

2.1. DEFINITION. We say that a pointed topological space $(X, *)$ is a *D-space* if it is a D-object in the category Sh_*.

Obviously, every object of HPol_* is a D-space.

2.2. EXAMPLES. (i) Let $(\Sigma, *)$ be the pointed Warsaw circle. This is the limit of an inverse sequence

$$(\boldsymbol{X}, *) = \big((X_n, *), p_{nn+1}\big)$$

with $X_n = S^1$ and each $p_{nn+1}\colon (X_{n+1}, *) \to (X_n, *)$ being an H-map of degree one. For each $m \geqslant n$, p_{mn} is a homotopy equivalence. Therefore $(\Sigma, *)$ is a D-space.

(ii) Let $(H, *)$ be the Hawaiian earring. This is the inverse limit of an inverse sequence

$$(S^1, *) \xleftarrow{p_{12}} (S^1 \vee S^1, *) \xleftarrow{p_{23}} (S^1 \vee S^1 \vee S^1, *) \leftarrow \ldots$$

for which the inclusion map

$$p^*_{nn+1}\colon (\underbrace{S^1 \vee S^1 \vee \ldots \vee S^1}_{n}, *) \to (\underbrace{S^1 \vee S^1 \vee \ldots \vee S^1}_{n+1}, *)$$

satisfies $p_{nn+1}\, p^*_{nn+1} = 1$. Thus $(H, *)$ is a D-space.

(iii) Let $(\boldsymbol{X}, *) = \big((X_n, *), p_{nn+1}\big)$ be an arbitrary inverse sequence in Top_*. If X_n are polyhedra, then the Overton–Segal star construction $(X^*, *)$ (see [11]) is a D-space.

Using Proposition 1.3 we obtain

2.3. PROPOSITION. *Every D-space $(X, *)$ is a pointed movable space.*

The next result is obtained from Theorem 1.5.

2.4. THEOREM. *The property of a pointed topological space to be a D-space is shape invariant.*

2.5. EXAMPLE. Every stable pointed space is a D-space. The converse is false: for example, the pointed Hawaiian earring is a D-space but it is not stable (see [9], p. 185).

3. Evaluation pro-groups and shape evaluation groups. For a pointed topological space $(X, *)$ denote by $G_n(X, *)$ the n-th evaluation subgroup of $\pi_n(X, *)$. Recall from [7] the definition of $G_n(X, *)$.

Let S^n be the n-sphere. Consider the continuous maps $F\colon X \times S^n \to X$ such that $F(x, s_0) = x$, where $x \in X$ and s_0 is the base point of S^n. Then the map $f\colon (S^n, s_0) \to (X, *)$ defined by $f(s) = F(*, s)$ represents an element $\alpha = [f] \in \pi_n(X, *)$. The set of all elements $\alpha \in \pi_n(X, *)$ obtained in this manner from some F determines the subgroup $G_n(X, *)$.

Not all results about homotopy groups are preserved for the evaluation subgroups. For example, it is not true that $f\colon (X, *) \to (Y, *)$ induces a map from $G_n(X, *)$ to $G_n(Y, *)$ (see [4] and [5]). However, if f is a homotopy domination, it is true that

$$f_*\colon \pi_n(X, *) \to \pi_n(Y, *)$$

carries $G_n(X, *)$ into $G_n(Y, *)$ (see [7]). Then, in [7] it was shown that if X and Y are both of the homotopy type of a CW-complex and if f is a homotopy equivalence, then f carries $G_n(X, *)$ isomorphically onto $G_n(Y, *)$.

3.1. THEOREM. *Let $(X, *)$ be a D-space and let*

$$p\colon (X, *) \to (\boldsymbol{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$$

be an HPol_**-expansion with $(\boldsymbol{X}, *)$ a D-system. Consider the homomorphism*

$$(p_{\lambda\lambda'})_*\colon \pi_n(X_{\lambda'}, *) \to \pi_n(X_\lambda, *) \quad \textit{for } n \geq 1 \textit{ and } \lambda \leq \lambda'.$$

Then the following pro-group is well defined:

$$\text{pro-}G_n(X, *) = (G_n(X_\lambda, *), (p_{\lambda\lambda'})_*, \Lambda)$$

and it is a subobject of pro-$\pi_n(X, *)$. *This pro-group depends on $(X, *)$ up to a natural isomorphism of pro-groups.*

Proof. For $\lambda \leq \lambda'$ the bonding map

$$p_{\lambda\lambda'}\colon (X_{\lambda'}, *) \to (X_\lambda, *)$$

is a homotopy domination. By [7] it is induced by the homomorphism

$$(p_{\lambda\lambda'})_*\colon G_n(X_{\lambda'}, *) \to G_n(X_\lambda, *).$$

Clearly, $(G_n(X_\lambda, *), (p_{\lambda\lambda'})_*, \Lambda)$ is a pro-group.

If we consider the inclusions

$$i_\lambda\colon G_n(X_\lambda, *) \to \pi_n(X_\lambda, *) \quad \text{for every } \lambda \in \Lambda,$$

then $i = (i_\lambda)$ is a monomorphism of pro-groups.

Now, if $q\colon (X, *) \to (\mathbf{Y}, *) = ((Y_\mu, *), q_{\mu\mu'}, M)$ is another HPol_*-expansion of $(X, *)$, with $(\mathbf{Y}, *)$ a D-system, then there exists a unique isomorphism $j\colon (\mathbf{X}, *) \to (\mathbf{Y}, *)$ in the category pro-HPol_* such that $jp = q$. Then j induces an isomorphism of pro-groups

$$j_*(j_{\lambda\mu})\colon (G_n(X_\lambda, *), (p_{\lambda\lambda'})_*, \Lambda) \to (G_n(Y_\mu, *), (q_{\mu\mu'})_*, M).$$

Therefore, one can assign to the D-space $(X, *)$ the equivalence class of pro-groups which contains $(G_n(X_\lambda, *), (p_{\lambda\lambda'})_*, \Lambda)$. Denote this class by pro-$G_n(X, *)$.

3.2. Definition. The pro-group pro-$G_n(X, *)$ is called the *n-th evaluation pro-group of the D-space* $(X, *)$.

The limit $\check{G}_n(X, *) = \lim \text{pro-}G_n(X, *)$ is called the *n-th shape evaluation group of the D-space* $(X, *)$.

The monomorphism of pro-groups

$$i\colon \text{pro-}G_n(X, *) \to \text{pro-}\pi_n(X, *)$$

induces a monomorphism of groups $\check{i}\colon \check{G}_n(X, *) \to \check{\pi}_n(X, *)$.

Clearly, $\check{G}_n(X, *)$ is defined up to a natural isomorphism of groups.

If $(X, *) \in \mathrm{HPol}_*$, then pro-$G_n(X, *)$ is a rudimentary pro-group and $\check{G}_n(X, *) = G_n(X, *)$. But, generally, this equality is false since just $\check{\pi}_n(X, *)$ differs from $\pi_n(X, *)$.

3.3. Theorem. *Let* $(X, *)$ *and* $(Y, *)$ *be two D-spaces. If* $F\colon (X, *) \to (Y, *)$ *is a pointed shape domination, then F induces a natural homomorphism of pro-groups*

$$\text{pro-}G_n(F)\colon \text{pro-}G_n(X, *) \to \text{pro-}G_n(Y, *)$$

and a natural homomorphism of groups

$$\check{G}_n(F)\colon \check{G}_n(X, *) \to \check{G}_n(Y, *).$$

Proof. Let $G\colon (Y, *) \to (X, *)$ be a right shape inverse of F and let

$$p\colon (X, *) \to (\mathbf{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda), \qquad q\colon (Y, *) \to (\mathbf{Y}, *) = ((Y_\mu, *), q_{\mu\mu'}, M)$$

be HPol_*-expansions such that $(\mathbf{X}, *)$ and $(\mathbf{Y}, *)$ are D-systems.

Consider

$$\mathbf{F} = (f_\mu, \varphi)\colon (\mathbf{X}, *) \to (\mathbf{Y}, *), \qquad \mathbf{G} = (g_\lambda, \psi)\colon (\mathbf{Y}, *) \to (\mathbf{X}, *),$$

two morphisms in pro-HPol_* defining the morphisms F and G, respectively, i.e., $q F = \mathbf{F} p$ and $pG = \mathbf{G} q$. The relation $\mathbf{F}\mathbf{G} = 1_{\mathbf{Y}}$ implies that each $\mu \in M$

admits $\mu' \in M$, $\mu' \geqslant \psi\varphi(\mu)$, $\mu' \geqslant \mu$ such that

$$f_\mu g_{\varphi(\mu)} q_{\psi\varphi(\mu)\mu'} = q_{\mu\mu'}.$$

Consequently, since $q_{\mu\mu'}$ has a right homotopy inverse $q^*_{\mu\mu'}$, we obtain

$$f_\mu [g_{\varphi(\mu)} q_{\psi\varphi(\mu)\mu'} q^*_{\mu\mu'}] = 1_{Y_\mu},$$

which shows that f_μ is a homotopy domination. Then, by [7], Propositions 1–4, we deduce that the homomorphism

$$(f_\mu)_*\colon \pi_n(X_{\varphi(\mu)}, *) \to \pi_n(Y_\mu, *)$$

carries $G_n(X_{\varphi(\mu)}, *)$ into $G_n(Y_\mu, *)$. Thus, we obtain a homomorphism of pro-groups

$$\text{pro-}G_n(F) = ((f_\mu)_*, \varphi)\colon \text{pro-}G_n(X, *) \to \text{pro-}G_n(Y, *).$$

The homomorphism $\check{G}_n(F)$ is the inverse limit $\lim \text{pro-}G_n(F)$.

3.4. Corollary. *If $(X, *)$ and $(Y, *)$ are D-spaces and $F\colon (X, *) \to (Y, *)$ is a pointed shape equivalence, then*

$$\text{pro-}G_n(F)\colon \text{pro-}G_n(X, *) \to \text{pro-}G_n(Y, *), \quad \check{G}_n(F)\colon \check{G}_n(X, *) \to \check{G}_n(Y, *)$$

are isomorphisms.

3.5. Theorem. *Let $F\colon (X, *) \to (Y, *)$ be a pointed shape morphism between two D-spaces such that F has a left shape inverse*

$$F'\colon (Y, *) \to (X, *)$$

and consider the homomorphism

$$\check{\pi}_n(F)\colon \check{\pi}_n(X, *) \to \check{\pi}_n(Y, *).$$

Then $\check{\pi}_n(F)(\alpha) \in \check{G}_n(Y, *)$ implies $\alpha \in \check{G}_n(X, *)$.

Proof. With the same notation as in Theorem 3.3, each λ admits $\lambda' \in \Lambda$, $\lambda' \geqslant \varphi\psi(\lambda)$, $\lambda' \geqslant \lambda$, such that

$$g_\lambda f_{\psi(\lambda)} p_{\varphi\psi(\lambda)\lambda'} = p_{\lambda\lambda'}.$$

Since $(X, *)$ is a D-system, g_λ is a homotopy domination. Using [7], Propositions 1–4, we obtain a homomorphism of groups

$$\check{G}_n(F') = \lim \text{pro-}G_n(F') = \check{\pi}_n(F')/\check{G}_n(Y, *)\colon \check{G}_n(Y, *) \to \check{G}_n(X, *).$$

If $\check{\pi}_n(F)(\alpha) = \beta \in \check{G}_n(Y, *)$, we can write

$$\alpha = \check{\pi}_n(F' F)(\alpha) = \check{\pi}_n(F')\check{\pi}_n(F)(\alpha) = G_n(F')(\beta) \in \check{G}_n(X, *)$$

for $\beta = \check{\pi}_n(F)(\alpha)$.

3.6. Theorem. *Let $(X, *)$ be a D-space and let*

$$p = (p_\lambda)\colon (X, *) \to (\boldsymbol{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$$

be an HTop$_*$*-expansion such that for every* λ *the H-map*

$$p_\lambda\colon (X, *) \to (X_\lambda, *)$$

is a homotopy domination and $(X_\lambda, *)$ *is a D-space. Then there exist an inverse system of groups*

$$\check{G}_n(X, *) = (\check{G}_n(X_\lambda, *), \check{G}_n(p_{\lambda\lambda'}), \Lambda)$$

and a natural isomorphism

$$\check{G}_n(X, *) \cong \lim \check{G}_n(X, *).$$

Proof. Under the imposed conditions there exist evaluation shape groups $\check{G}_n(X, *)$, $\check{G}_n(X_\lambda, *)$, $\lambda \in \Lambda$, and homomorphisms

$$\check{G}_n(p_\lambda)\colon \check{G}_n(X, *) \to \check{G}_n(X_\lambda, *).$$

Moreover, by the relation $p_{\lambda\lambda'} p_{\lambda'} = p_\lambda$, $\lambda \leqslant \lambda'$, and since p_λ is a homotopy domination, it follows that so is also $p_{\lambda\lambda'}$. In this way we obtain an inverse system of groups

$$\check{G}_n(X, *) = (\check{G}_n(X_\lambda, *), \check{G}_n(p_{\lambda\lambda'}), \Lambda)$$

and a homomorphism of pro-groups

$$\check{G}_n(p) = (\check{G}_n(p_\lambda))\colon \check{G}_n(X, *) \to \check{G}_n(X, *).$$

Therefore, a homomorphism of groups is defined:

$$\lim \check{G}_n(p)\colon \check{G}_n(X, *) \to \lim \check{G}_n(X, *);$$

namely,

$$\lim \check{G}_n(p) = \lim_\lambda \check{\pi}_n(p_\lambda)/\check{G}_n(X, *).$$

It follows ([9], Theorem 7, p. 130) that $\lim \check{G}_n(p)$ is a monomorphism.

Now, let $q = (q_\mu)\colon (X, *) \to (Y, *) = ((Y_\mu, *), q_{\mu\mu'}, M)$ be an HPol$_*$-expansion with $(Y, *)$ a D-system in HPol$_*$. There exist ([9], I, Section 4.1) two morphisms

$$f\colon (X, *) \to (Y, *) \quad \text{and} \quad g\colon (Y, *) \to (X, *)$$

in pro-HTop$_*$ such that $fg = 1_Y$ and g induces $\lim \check{G}_n(p_\lambda)$. Applying the fact that $\lim_\lambda \check{\pi}_n(p_\lambda)$ is an epimorphism ([9], Theorem 7, p. 130) and using Theorem 3.5 we see that $\lim \check{G}_n(p_\lambda)$ is also an epimorphism.

3.7. Corollary. *If* $(X, *)$ *is a D-space for which there exists an* HPol$_*$*-expansion* $p\colon (X, *) \to (X, *)$ *with* $(X, *)$ *a D-sequence* $((X_n, *), p_{nn+1})$ *in* HPol$_*$, *then* $\check{G}_n(X, *) = 0$ *iff* pro-$G_n(X, *) \cong 0$ *in* pro-Grp.

3.8. Examples. (i) Let $(\Sigma, *)$ be the pointed Warsaw circle. Then

$$\text{pro-}G_n(\Sigma, *) = G_n(S^1, *) = 0, \qquad n \geqslant 2,$$

and

$$\text{pro-}G_1(\Sigma, *) = G_1(S^1) \cong Z$$

(see [5], Theorem 5.4).

(ii) Let $(H, *)$ be the pointed Hawaiian earring. Clearly,

$$\text{pro-}G_n(H, *) = 0 \quad \text{for } n \geq 2.$$

For $n = 1$ we have (see [7], Corollary 2.4)

$$G_1(X_m, *) \subseteq Z(\pi_1(X_m, *)),$$

the center of

$$\pi_1(X_m, *) = \underbrace{Z * \ldots * Z}_{m}.$$

Thus, $G_1(X_1, *) = Z$ and $G_1(X_m, *) = 0$ if $m \geq 2$. In conclusion, for $n \geq 1$, $\check{G}_n(H, *) = 0$.

4. ε-cross-sections for approximate fibrations.

4.1. The approximate fibrations were introduced and studied by Coram and Duvall ([2], [3]) by considering the following approximate homotopy lifting property (AHLP). A map $p\colon E \to B$ between compact metric ANR's has AHLP with respect to a space X provided that for every $\varepsilon > 0$ and for each map $h\colon X \to E$ and each homotopy $H\colon X \times I \to B$, with $ph = H_0$, there exists a homotopy $\tilde{H}\colon X \times I \to E$ satisfying $\tilde{H}_0 = h$ and $d(p\tilde{H}, H) < \varepsilon$. For an approximate fibration $p\colon E \to B$ we obtain an exact sequence of groups:

$$\ldots \to \check{\pi}_k(F, *) \xrightarrow{i_*} \pi_k(E, *) \xrightarrow{p_*} \pi_k(B, *) \xrightarrow{d} \check{\pi}_{k-1}(F, *) \to \ldots,$$

where $F = p^{-1}(*)$ is the fiber over the base point and $i\colon (F, *) \to (E, *)$ is the inclusion map.

4.2. THEOREM. *Let $p\colon E \to S^n$ be an approximate fibration, $n \geq 2$. If the fiber $(F, *)$ is a D-space, then $d(\pi_n(S^n, *)) \subseteq \check{G}_{n-1}(F, *)$.*

Proof. This theorem is a generalization of Gottlieb's theorem ([7], Theorem 2.6) for Hurewicz fibrations, whose proof is based on the Stasheff's classification theorem [13] (see also [6]). Unfortunately, a similar classification theorem for approximate fibrations is not known. However, we can prove our theorem using the s-fibrations [8].

Recall that an s-fibration is a morphism

$$\mathbf{p} = (p_\lambda)\colon \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda) \to B$$

in pro-Top, where Λ is a cofinite and directed set, B is an ANR and satisfies the following form of HLP. Given $\lambda \in \Lambda$ there exists $\lambda' \geq \lambda$ such that, whenever X is a topological space and $g\colon X \to E_\lambda$ and $H\colon X \times I \to B$ are maps with $p_\lambda g = H_0$, there exists $G\colon X \times I \to E_{\lambda'}$ so that $p_{\lambda'} G = H$ and $q_{\lambda\lambda'} G_0 = g$. If $x \in B$, the fiber of $\mathbf{p}$ at x is the inverse system

$$F = p^{-1}(x) = (p^{-1}(x), q_{\lambda\lambda'}/p^{-1}(x), \Lambda).$$

To the given approximate fibration $p\colon E \to S^n$ one can associate ([8], Theorem 3.1) an s-fibration

$$\boldsymbol{p} = (p_i)\colon\ \boldsymbol{E} = (E_i, q_{ij}, N) \to S^n.$$

Coram and Duvall established [2] that the fibers of an approximate fibration are FANR's. Then, since $(F, *)$ is a D-space, we can suppose that all pointed fibers $(p_i^{-1}(*), *)$ are D-spaces. Now, to the s-fibration $\boldsymbol{p}$ there corresponds ([8], Theorem 11.1) a bundle equivalence

$$f\colon\ \boldsymbol{F} \times S^{n-1} \to \boldsymbol{F} \times S^{n-1}$$

such that $f/\boldsymbol{F} \times s_0 = 1_{\boldsymbol{F} \times s_0}$. By composing with the first projection and using Theorem 3.6, we obtain an element $[\mathrm{pr}_1 \circ f] \in \check{G}_{n-1}(F, *)$. We shall prove that

$$[\mathrm{pr}_1 \circ f] = d([1_{S^n}]).$$

In fact, using the covering homotopy theorem ([8], Theorem 5.3) we obtain an element $u \in \check{\pi}_n(\boldsymbol{E}, \boldsymbol{F}, *)$ for which $\boldsymbol{p}_*(u) = [1_{S^n}]$, where $\boldsymbol{p}_*$ is the isomorphism

$$\boldsymbol{p}_*\colon\ \check{\pi}_n(\boldsymbol{E}, \boldsymbol{F}, *) \cong \pi_n(S^n, *).$$

Then $d([1_{S^n}]) = d(\boldsymbol{p}_*(u))$ and by [2], Corollary 3.5, we get the equality

$$d([1_{S^n}]) = \delta \boldsymbol{p}_*^{-1} \boldsymbol{p}_*(u) = \delta(u),$$

where $\delta\colon \check{\pi}_n(\boldsymbol{E}, \boldsymbol{F}, *) \to \check{\pi}_{n-1}(\boldsymbol{F}, *)$ is the usual boundary homomorphism. But $\delta(u) = [\mathrm{pr}_1 \circ f]$, so that we obtain $[\mathrm{pr}_1 \circ f] = d([1_{S^n}])$, which implies the inclusion

$$d(\pi_n(S^n, *)) \subseteq \check{G}_{n-1}(F, *).$$

4.3. We recall from [12] that an *ε-cross-section* (for $\varepsilon > 0$) of an approximate fibration $p\colon E \to B$ is a map $s\colon B \to E$ such that $d(ps, 1_B) < \varepsilon$. If p has an ε-cross-section for each $\varepsilon > 0$, we say that p has *approximate cross-sections.*

4.4. COROLLARY. *Let $p\colon E \to S^n$ be an approximate fibration. If the fiber $(F, *)$ is a D-space such that $\check{G}_{n-1}(F, *) = 0$, then p has approximate cross-sections.*

Proof. Using the exact sequence of p and Theorem 4.2, we obtain

$$\mathrm{Im}\, d \subseteq \check{G}_{n-1}(F, *) = 0,$$

which implies $\mathrm{Ker}\, d = \pi_n(S^n, *)$, and therefore $p_*\colon \pi_n(E, *) \to \pi_n(S^n, *)$ is surjective. There exist

$$s'\colon (S^n, *) \to (E, *) \quad \text{and} \quad H\colon (S^n \times I, * \times I) \to (S^n, *)$$

such that $H\colon ps' \simeq 1_{S^n}$. Then for every $\varepsilon > 0$ there exists $\tilde{H}\colon S^n \times I \to E$ with $\tilde{H}_0 = s'$ and $d(p\tilde{H}, H) < \varepsilon$. Consider $s\colon S^n \to E$ defined by $s = \tilde{H}_1$. Then

$$d(ps(x), x) = d(p\tilde{H}(x, 1), H(x, 1)) < \varepsilon \quad \text{for every } x \in S^n.$$

Therefore $d(ps, 1_{S^n}) < \varepsilon$. Thus s is an ε-cross-section.

4.5. EXAMPLE. Let $(H, *)$ be the pointed Hawaiian earring and let $p\colon E \to S^n$ be an approximate fibration with the fiber $(H, *)$. Then, for $n \geq 2$, p has approximate cross-sections.

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