COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 2

# ON m-CONVEX $B_0$-ALGEBRAS OF TYPE ES

BY

W. ŻELAZKO (WARSZAWA)

A $B_0$-*algebra* is a completely metrisable locally convex topological algebra. The topology of a $B_0$-algebra $A$ can be given by means of an increasing sequence of pseudonorms

(1) $$\|x\|_i \leqslant \|x\|_{i+1},$$

$i = 1,2,\ldots,$ $x \in A$, which satisfies

(2) $$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1},$$

$i = 1,2,\ldots,$ $x,y \in A$.

A $B_0$-algebra $A$ is said to be *multiplicatively convex*, or *m-convex*, if the pseudonorms (1) can be chosen in such a way that

(3) $$\|xy\|_i \leqslant \|x\|_i\|y\|_i,$$

$i = 1,2,\ldots,$ $x,y \in A$. In this case to each index $i$ there corresponds a closed ideal

$$N_i = \{x \in A: \|x\|_i = 0\},$$

and a Banach algebra $A_i$ which is the completion of $A/N_i$ in the norm $\|x\|_i$. If $\pi_i$ denotes the natural projection of $A$ into $A_i$, then the spectrum $\sigma(x)$ of an element $x \in A$ is given by

(4) $$\sigma(x) = \bigcup_{i=1}^{\infty} \sigma_i(x),$$

where $\sigma_i(x) = \sigma_{A_i}(\pi_i(x))$ (cf. [1] or [3]). The sets $\sigma_i(x)$ are compact subsets of the complex plane and

(5) $$\sup|\sigma_i(x)| \leqslant \|x\|_i.$$

In paper [4] we introduced a concept of a commutative complex Banach algebra of type *ES* (from *Extensions from Subalgebras*). We write namely that $A \in ES$ if for every closed subalgebra $A_0$ of it every multiplicative linear functional defined on $A_0$ can be extended to such a functional on $A$. We have proved that $A \epsilon ES$ if and only if for every element $x \epsilon A$ its spectrum is totally disconnected (in the case of a compact set this is equivalent with the statement that it contains no continuum). In paper [5] we extended the concept of an $ES$-algebra to the non-commutative case and we proved that the class of $ES$-algebras is closed with respect to taking subalgebras, quotient algebras, finite direct products and homomorphic images.

In this paper we obtain similar results for (complex) $m$-convex $B_0$-algebras, and so we answer a question posed in [5]. Our main result is contained in Theorem 1. We show also that this result is no longer true if we drop the assumption of metrisability and assume merely the $m$-convexity and completeness.

We assume throughout this paper that every considered algebra is an algebra over the field of complex numbers.

Definition 1. Let $A$ be a commutative topological algebra. The algebra $A$ is said *to be of type ES* (*to be an ES-algebra* or *to belong to the class ES*, written $A \epsilon ES$) if for every closed subalgebra $A_0 \subset A$ every continuous multiplicative linear functional defined on $A_0$ is extendable to such a functional defined on $A$.

LEMMA 1. *Let $A$ be a commutative complex $B_0$-algebra with unit $e$. Suppose that for some $x_0 \epsilon A$ the spectrum $\sigma(x_0)$ contains a continuum $K$. Then $A$ is not an ES-algebra.*

Proof. By formula (4) we have

$$K = K \cap \sigma(x_0) = \bigcup_{i=1}^{\infty} [K \cap \sigma_i(x_0)]$$

and by category arguments there is an index $i_0$ such that $K \cap \sigma_i(x_0)$ contains a continuum $K_0$. We have then $K_0 \subset \sigma_{i_0}(x_0)$.

Let $\alpha, \beta \epsilon K_0$, $\alpha \neq \beta$. It is clear that if we put

$$y = \frac{\pi}{2} \frac{x_0 - \alpha e}{\beta - \alpha},$$

then we obtain an element with spectrum $\sigma_{i_0}(y)$ containing a continuum $K_1$ such that $0, \frac{1}{2}\pi \epsilon K_1$. So if we set

$$z = \exp(iy),$$

we see that $\sigma_{i_0}(z)$ contains a continuum $K_2$ connecting 1 and $i$, and $z$ is an invertible element in $A$. It is easy to see that $\sigma_{i_0}(z^4)$ separates the complex plane between 0 and $\infty$. Let $A_0$ be the subalgebra of $A$ generated by $z^4$ and containing the unit $e$. The elements of the form $p(z^4)$, where $p$ is a polynomial with complex coefficients, form a dense subalgebra of $A_0$. For every such an element we put

$$f(p(z^4)) = p(0),$$

and so we obtain a multiplicative and linear functional defined on a dense subalgebra of $A_0$. The functional $f$ is moreover, a continuous functional since, by the maximum principle and formula (5), we have

$$\begin{aligned}|f(p(z^4))| &= |p(0)| \leqslant \max_{t\in\sigma_{i_0}(z^4)} |p(t)| \\ &= \max|\sigma_{i_0}[p(z^4)]| \leqslant \|p(z^4)\|_{i_0}.\end{aligned}$$

So $f$ can be extended, by the continuity, onto the whole of $A$. On the other hand, $f$ cannot be extended to a multiplicative linear functional defined on $A$, since $f(z^4) = 0$, and $z^4$ is an invertible element in $A$. Thus $A$ is not an $ES$-algebra.

LEMMA 2. *Suppose that* $1^0$ *$A$ is a commutative Banach algebra with unit $e$, and* $2^0$ *$A$ contains a dense set $A_0 \ni e$ such that for every $x \in A_0$ its spectrum $\sigma(x)$ does not contain any continuum. Then the maximal ideal space $\mathfrak{M}(A)$ coincides with the Shilov boundary $\Gamma(A)$.*

Proof. Let $M_0 \in \mathfrak{M}(A)$. Let $U$ be a neighbourhood of $M_0$ of the form

$$\{M \in \mathfrak{M}(A)\colon |x_i^\wedge(M)| < \varepsilon,\ i = 1, 2, \ldots, n\}, \tag{6}$$

where $x_1, \ldots, x_n \in M_0 \cap A_0$ and $x^\wedge$ denotes the Gelfand transform of an element $x$. Since $e \in A_0$ and $A_0$ is dense in $A$, the neighbourhoods (6) form a basis of neighbourhoods of the ideal $M_0$. Consider the joint spectrum

$$\sigma(x_1, \ldots, x_n) = \{(x_1^\wedge(M), \ldots, x_n^\wedge(M)) \in C^n\colon M \in \mathfrak{M}(A)\},$$

where $x_1, \ldots, x_n$ are elements defining the neighbourhood (6). Since $\sigma(x_k)$, $1 \leqslant k \leqslant n$, is the projection of $\sigma(x_1, \ldots, x_n)$ onto the $k$-th coordinate plane, it follows that the joint spectrum $\sigma(x_1, \ldots, x_n)$ also contains no continuum or, equivalently, it is totally disconnected. It follows that the intersection

$$\sigma(x_1, \ldots, x_n) \cap \{(t_1, \ldots, t_n) \in C^n\colon |t_k| < \varepsilon,\ k = 1, \ldots, n\} \tag{7}$$

contains a non-void open and closed subset $S$ containing the origin $(0, \ldots, 0)$, and so there is in $C^n$ an open neighbourhood $V$ of the origin such that $S = V \cap \sigma(x_1, \ldots, x_n)$, and there is also an open $W \subset C^n$ such that $\sigma(x_1, \ldots, x_n) \setminus S = W \cap \sigma(x_1, \ldots, x_n)$. We put

$$\varphi(t_1, \ldots, t_n) = \begin{cases} 1 & \text{for } (t_1, \ldots, t_n) \in V, \\ 0 & \text{for } (t_1, \ldots, t_n) \in W. \end{cases}$$

This is an analytic function defined on the open set $V \cup W$ containing the joint spectrum $\sigma(x_1, \ldots, x_n)$. By the operational calculus of several complex variables (cf. e.g. [2]) applied to $\varphi$ and $x_1, \ldots, x_n$ there is an element $y \in A$ such that $y^\wedge(M) = 1$ if $(x_1^\wedge(M), \ldots, x_n^\wedge(M)) \in S$ and $y^\wedge(M) = 0$ otherwise. Since $S$ is contained in the set (7), it follows that $y^\wedge(M)$ assumes its maximal value in the neighbourhood U given by (6). Since neighbourhoods (6) form a basis for $M_0$, this implies $M_0 \in \Gamma(A)$, and so $\mathfrak{M}(A) = \Gamma(A)$.

LEMMA 3. *If $A$ is a commutative m-convex $B_0$-algebra with unit $e$, and for every $x \in A$ the spectrum $\sigma(x)$ contains no continuum, then $A$ is an ES-algebra.*

Proof. Let $A_0$ be a closed subalgebra of $A$ containing the unit $e$. To a system (1) in $A$ there corresponds a sequence $A_i$ of Banach algebras, and to the same system in $A_0$ there corresponds a sequence $A_i^0$. We clearly have $A_i^0 \subset A_i$, and, moreover, the natural projection of $A_0$ into $A_i^0$ is the restriction to $A_0$ of the natural projection of $A$ into $A_i$. Let $f$ be a continuous multiplicative linear functional defined on $A_0$. There exists an index $i_0$ such that

$$|f(x)| \leqslant \|x\|_{i_0}$$

for every $x \in A_0$, and so there is a multiplicative linear functional $f_0$ defined on $A_{i_0}^0$ such that

$$f_0[\pi_{i_0}(x)] = f(x), \quad x \in A_0. \tag{8}$$

Since $\sigma(x)$ contains no continuum and because of (4), $\sigma_{i_0}(x)$ contains a continuum for no $x \in A$. Consequently, (cf. e.g. [2])

$$\sigma_{A_{i_0}}(x) = \sigma_{A_{i_0}^0}(x)$$

for every $x \in A_0$. We can now apply Lemma 2 taking the Banach algebra $A_{i_0}^0$ as $A$ and the dense subalgebra $\pi_{i_0}(A_0)$ as the set $A_0$, and so we can extent the functional $f_0$ to a multiplicative and linear functional $F_0$ defined on $A_{i_0}$. If we put

$$F(x) = F_0[\pi_{i_0}(x)],$$

we obtain a continuous multiplication linear functional on $A$ which is clearly an extension of the functional $f$, and so Lemma 3 holds.

If $A$ is an m-convex algebra without unit element, then it may be imbedded in an algebra $A_1$ with a unit, and $A$ is a maximal ideal in $A_1$ of codimension 1. Since for the spectra we have

$$\sigma_A(x) = \sigma_{A_1}(x), \quad x \in A,$$

and since every continuous multiplicative linear functional on $A$ can be extended to such a functional on $A_1$, Lemma 1 and Lemma 3 imply

THEOREM 1. *A commutative m-convex $B_0$-algebra is an ES-algebra if and only if the spectrum of any of its elements contains no continuum.*

If $A$ is an algebra with a unit, then $G(A)$ denotes the group of all invertible elements in $A$. In a similar way as in [5] we can prove the following

THEOREM 2. *Let $A$ be an m-convex $B_0$-algebra with a unit element. Then the following statements are equivalent:*

(i) *Every commutative (closed) subalgebra of $A$ is an ES-algebra.*

(ii) *For every $x \in A$ its spectrum contains no continuum.*

(iii) *For every (closed) subalgebra $A_0 \subset A$, containing the unit element we have*

$$G(A_0) = G(A) \cap A_0.$$

This theorem gives a motivation to the following

Definition 2. Let A be a $B_0$-algebra with a unit element. It is called an *ES-algebra* if any one of the three equivalent conditions (i)-(iii) of Theorem 2 holds true.

Denoting by $Q(A)$ the group of all quasiregular elements of the algebra $A$ (cf. e.g. [2]) we can rewrite Theorem 2 as

THEOREM 3. *Let $A$ be an m-convex $B_0$-algebra. Then the following three conditions are equivalent:*

(i) *and* (ii) *the same as in Theorem* 2.

(iii') *For every (closed) subalgebra $A_0 \subset A$ we have*

$$Q(A_0) = Q(A) \cap A_0.$$

As before, this theorem gives a motivation to the following more general

Definition 3. An $m$-convex $B_0$-algebra $A$ is said to be an *ES-algebra* if any one of the three equivalent conditions of Theorem 3 holds true.

The following theorem shows that the class of $m$-convex $B_0$-algebras of type *ES* is closed with respect to natural algebraic operations. Its proof is exactly the same as in [5].

THEOREM 4. *Assume $A$ to be an m-convex $B_0$-algebra of type ES. Then*

(i) *If $A_0$ is a closed subalgebra of $A$, then $A_0 \in ES$.*

(ii) *If $I$ is a closed two-sided ideal in $A$, then $A/I \in ES$.*

(iii) *If $A_0$ is an m-convex $B_0$-algebra and there is a homomorphism of $A$ onto $A_0$, then $A_0 \in ES$.*

(iv) *The cartesian product of a finite number of m-convex $B_0$-algebras of type ES is again such an algebra.*

To close this paper, we give an example of a complete $m$-convex locally convex algebra of type $ES$ (according to definition 1) containing elements with arbitrary spectra.

Example. Let $J$ be an index set of continuum power and put $A = C^J$. $A$ is clearly a complete $m$-convex algebra with coordinatewise algebraic operations and with the product topology. It is clear that any non-void subset of the complex plane is the spectrum for some element in $A$. Let $A_0$ be a closed subalgebra of $A$ and let $f$ be a continuous multiplicative linear functional on $A_0$. The functional $f$ must be continuous with respect to a pseudonorm equal to the maximum of the absolute values of a finite number of coordinates, and so $f$ must be of the form $f(x) = x_j$ for some $j \in J$ ($x_j$ denotes the $j$-th coordinate of $x$). Consequently, $A$ is an $ES$-algebra.

## *REFERENCES*


[1] E. Michael. *Locally multiplicatively-convex topological algebras,* Memoirs of the American Mathematical Society 11 (1952).

[2] C. E. Rickart, *General theory of Banach algebras,* Princeton 1960.

[3] W. Żelazko, *Metric generalizations of Banach algebras,* Rozprawy Matematyczne 47 (1965).

[4] — *Concerning extension of multiplicative linear functionals in Banach algebras,* Studia Mathematica 31 (1968), p. 495-499.

[5] — *Concerning non-commutative Banach algebras of type ES,* Colloquium Mathematicum 20 (1968), p. 121-126.


INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

*Reçu par la Rédaction le 29. 2. 1968*