COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

FINITELY ADDITIVE INVARIANT MEASURES. II

BY

J. M. ROSENBLATT (COLUMBUS, OHIO)

This paper is a continuation of part I [7]. Our purpose is to show the existence of an interesting ideal in the algebra of Borel subsets of the sphere which is invariant under isometries and such that the sphere admits paradoxical decompositions with Borel parts modulo this ideal.

As in part I, we begin with more general algebraic facts. Let $\boldsymbol{B}$ be a Boolean algebra and let G be a group of automorphisms of $\boldsymbol{B}$. Assume that $A \subset \boldsymbol{B}$ is a set such that the elements $g(a)$ are different for different pairs $g \in G$ and $a \in A$ and, moreover, the elements $g(a)$ are free generators of a free Boolean subalgebra of $\boldsymbol{B}$. For every $a \in A$, let $f_a: G \to \{0, 1\}$. By $\boldsymbol{B}_2$ we denote the two-element Boolean algebra with universe $\{0, 1\}$.

LEMMA. *Under the conditions above, there exists a homomorphism $h: \boldsymbol{B} \to \boldsymbol{B}_2$ such that*

$$h(g(a)) = f_a(g) \quad \text{for all } g \in G \text{ and } a \in A. \tag{1}$$

Proof. Let $\boldsymbol{B}_A$ be the subalgebra of $\boldsymbol{B}$ generated by the elements $g(a)$ with $g \in G$ and $a \in A$. Since these elements are free generators, it is clear that there exists a homomorphism $h_A: \boldsymbol{B}_A \to \boldsymbol{B}_2$ which satisfies (1). By the prime ideal theorem, h_A extends to a homomorphism $h: \boldsymbol{B} \to \boldsymbol{B}_2$ which satisfies (1).

Let $\boldsymbol{B}$, G, and A be as above, and suppose that A is infinite.

THEOREM. *The group G is not amenable if and only if there exists a proper G-invariant ideal $\boldsymbol{I}$ in $\boldsymbol{B}$ such that the unity of the factor algebra $\boldsymbol{B}/\boldsymbol{I}$ has a paradoxical decomposition relative to G. Equivalently, G is amenable if and only if, for every proper G-invariant ideal $\boldsymbol{I}$ in $\boldsymbol{B}$, there exists a finitely additive invariant probability measure on $\boldsymbol{B}$ which vanishes on the elements of $\boldsymbol{I}$.*

Proof. The second formulation of the theorem follows from the first one and a theorem of Tarski stated in part I. As for the first version of the theorem, the "if" assertion follows part I, 5.1. It remains to prove the "only if" assertion.

Let $(A_1, \ldots, A_m, B_1, \ldots, B_n, g_1, \ldots, g_m, h_1, \ldots, h_n)$ be a paradoxical decomposition of G, i.e. the sets $A_1, \ldots, A_m, B_1, \ldots, B_n$ are disjoint subsets of G and

$$G = g_1(A_1)\cup\ldots\cup g_m(A_m) = h_1(B_1)\cup\ldots\cup h_n(B_n).$$

Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be distinct elements of A. Define

$$f_{a_i}(g) = \begin{cases} 1 & \text{if } g \in A_i, \\ 0 & \text{if } g \notin A_i, \end{cases} \quad \text{and} \quad f_{b_i} = \begin{cases} 1 & \text{if } g \in B_i, \\ 0 & \text{if } g \notin B_i. \end{cases}$$

Let $h: \boldsymbol{B} \to \boldsymbol{B}_2$ be the homomorphism given by the Lemma for the system f_a with $a \in \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Let

$$\boldsymbol{I} = \{a \in \boldsymbol{B}: h(g(a)) = 0 \text{ for all } g \in G\}.$$

Then $\boldsymbol{I}$ is a G-invariant ideal, and it is not difficult to check that

$$(a_1/\boldsymbol{I}, \ldots, a_m/\boldsymbol{I}, b_1/\boldsymbol{I}, \ldots, b_n/\boldsymbol{I}, g_1, \ldots, g_m, h_1, \ldots, h_n)$$

constitutes a paradoxical decomposition of unity in $\boldsymbol{B}/\boldsymbol{I}$.

Application. Let $\boldsymbol{B}$ be the Boolean algebra of Lebesgue measurable subsets of S^2 (the 2-dimensional sphere in R^3) modulo sets of measure zero, and let G be the group of isometries of S^2. As mentioned in part I, G is not amenable and it acts as a group of automorphisms of $\boldsymbol{B}$. With the distance between elements in $\boldsymbol{B}$ being the measure of the symmetric difference, $\boldsymbol{B}$ is a complete metric space. Let $G^{(n)}$ be the set of n-tuples of different elements of G and, for $\bar{g} = (g_1, \ldots, g_n)$ in $G^{(n)}$ and $x \in \boldsymbol{B}$, let $\bar{g}(x) = (g_1(x), \ldots, g_n(x))$. Let R be any k-ary relation over $\boldsymbol{B}$ of the form

(2) there exist $\bar{g}_1, \ldots, \bar{g}_k \in G^{(n)}$ with

$$\varphi(\bar{g}_1(x_1), \ldots, \bar{g}_k(x_k)) = \boldsymbol{1},$$

where φ is any Boolean function which is not a tautology of the propositional calculus and $\boldsymbol{1}$ is the unity of $\boldsymbol{B}$. One can prove using the methods of [8] that R is of first category in $\boldsymbol{B}^k$ relative to the product topology. Then by the theorem of [4] (subsequently refined in [3] and [5]) there exists an infinite set $A \subset \boldsymbol{B}$ such that no distinct elements $x_1, \ldots, x_k \in A$ satisfy any relation of the form (2) for all possible values of k. In other words, the elements $g(a)$ with $g \in G$ and $a \in A$ constitute free generators of a free subalgebra of $\boldsymbol{B}$. Thus, the theorem applies and we get this interesting corollary:

COROLLARY. *There is an invariant ideal of Borel subsets of S^2 and a paradoxical decomposition of S^2 into Borel subsets modulo this ideal.*

However, the following long-standing problem of S. Ruziewicz [1] remains open:

Is the Lebesgue measure the only finitely additive invariant probability measure on $\boldsymbol{B}$?

E. Marczewski conjectured that there exists a proper invariant ideal $\boldsymbol{I} \subset \boldsymbol{B}$, containing the nowhere dense sets, and a finitely additive invariant probability measure on $\boldsymbol{B}$ which vanishes on the elements of $\boldsymbol{I}$.

Similar questions on unicity remain open for other group actions on compact spaces, in particular, a compact group acting on itself by left translation. But it is well known that the answer to the problem is negative if the group is amenable as a discrete group (see [1], [2], [6], [7], [9], [10], and [11]).

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Reçu par la Rédaction le 16. 2. 1978