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ON THE NILPOTENT ELEMENTS OF SEMIGROUPS

BY

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In a previous paper [3], the authors investigated the structure of algebraic radicals in semigroups and extended some results in commutative algebras to compact abelian semigroups with zero. In this paper *, we study the topological radical of a semigroup. Several results concerning abelian semigroups with zero and local zeros are obtained. This paper is independent of [3].

By the *topological radical* of a semigroup S with zero, we mean the union of all the nil ideals of S. An element b in S is called *nilpotent* if $b^n \to 0$, that is, if for every neighborhood U of 0, there exists an integer n_0 such that $b^n \in U$ for every $n \geqslant n_0$. We denote by N the set of all nilpotent elements of S. An ideal A of S consisting entirely of nilpotent elements is called a *nil ideal* of S. In case N is an ideal of S, N turns out to be our topological radical of S. The concept of nil ideals was first introduced by Numakura in 1951 [6]. In his paper, he investigated the structure of S when N is open. Some amplifications of his results on compact semigroups with zero were given by Koch [4].

It is the purpose of this paper to apply the work of Numakura [6] on topological semigroups with zero to compact abelian semigroups with zero and local zeros. We are mainly interested in studying some properties of the set N. We prove that if N can be embedded densely in an abelian semigroup S, then S has no local zeros. We also show that if a semigroup S contains zero and local zeros, then S must be disconnected. Some conditions which lead to the existence of a local zero in a compact N-semigroup are given. A characterization of compact abelian N-semigroups is obtained. Moreover, if S is a compact Ω-semigroup with zero such that $\bar{N}^2 \not\subset N$, then $\bar{N}-N$ is either a group or a semilattice of groups. The set of topological zero divisors of an element a in S will also be treated.

Throughout, for sets $X, Y \subseteq S$, $X-Y$ denotes the complement Y in X, XY denotes the set of all products xy with $x \in X$ and $y \in Y$, $\bar{X}$ denotes

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the closure of the set X in S and X' denotes the complement of X in S. All spaces are topological Hausdorff in this paper. Unless otherwise stated, S will always be regarded as a topological abelian semigroup with zero. The reader is referred to [7] for terminology and notation.

1. Definitions and preliminary results. In this section, N denotes the set of all nilpotent elements in S and S denotes a non-empty abelian semigroup with zero. To avoid trivialities, we suppose that $S \neq \{0\}$ and the space S has at least three points.

Notation. Let A be a subset of S.

$J(A) = A \cup AS$, that is, the smallest ideal containing A.

$J_0(A) =$ the union of all ideals contained in A, that is, the largest ideal contained in A if there are any.

Definition 1.1. S is said to be an *N-semigroup* if N is an open subset of S.

S is said to be an *Ω-semigroup* if for any two ideals I_1 and I_2 such that $I_1 \cap I_2 \neq \emptyset$, then either $I_1 \subset I_2$ or $I_2 \subset I_1$.

Definition 1.2. An element 0 such that $a0 = 0a = 0$ for all a in S is called a *zero element* of S and it is easily seen that 0 is uniquely defined.

An idempotent element $e \neq 0$ of S is called a *local zero* if there exists an open neighborhood U in S such that $e \in U$ and e is the zero for U, that is, $ex = xe = e$ for every $x \in U$. We observe that a zero is not a local zero.

Definition 1.3. Let a be an element of S. Define Tod $a = \{x \in S | ax \in N\}$ that is, the set of all topological zero divisors of S. Tod a is non-empty since $\{0\}$ is always in Tod a.

Definition 1.4. Let $k \geqslant 1$ be an integer. A *k-ideal* A of S is a non--vacuous subset of S such that $A^k S \subseteq A$. A *principal k-ideal* generated by any subset A of S is the set $J_k(A) = A \cup A^2 \cup \ldots \cup A^k S$, which is the smallest k-ideal generated by the sets $\{A, A^2, \ldots, A^k\}$.

The following results are elementary, but they are useful. Some important properties of the set N will be disclosed after these propositions and counter-examples.

PROPOSITION 1.5. (i) *The set N is always a subsemigroup of S.*

(ii) *Let S have a unit u. Then $u \in$ Tod a if and only if $a \in N$ for any $a \in S$.*

(iii) *If e is an idempotent element of S, then $N \subseteq$ Tod e.*

(iv) *If N is the topological radical of S, then* Tod e *is an ideal of S.*

(v) *If each* Tod a *is an ideal of S for every $a \in S$, then every principal k-ideal generated by an element $n \in N$ is contained in N for $k > 1$.*

(vi) *If N is connected and S contains at least an idempotent $e \neq 0$, then there exists a subsemigroup T of S such that every element of N is contained in a connected ideal of T.*

We only prove (v) and (vi); the others are direct consequences of our definitions and we leave them to the reader.

Proof. (v) Take $n \in N$, and let $k > 1$ be an integer. Then $n^k \in N$. Hence $n \in \mathrm{Tod}\, n^{k-1}$. By our assumption, $\mathrm{Tod}\, n^{k-1}$ is an ideal of S. Thus for any $y \in S$, $ny \in \mathrm{Tod}\, n^{k-1}$. That is, $n^k y \in N$. In other words, $n^k S \subseteq N$. Since the principal k-ideal generated by the element n is the set $J_k(n) = n \cup n^2 \cup \ldots \cup n^k \cup n^k S$, clearly, $J_k(n) \subseteq N$.

(vi) To prove (vi), we first note that $\mathrm{Tod}\, e$ is a subsemigroup of S. For let $x \in \mathrm{Tod}\, e$ and $y \in \mathrm{Tod}\, e$. Then we have $ex \in N$ and $ey \in N$. Since S is abelian, by (i), N is a subsemigroup of S. Thus $exy = e^2xy = (ex)(ey) \in N^2 \subset N$. Hence $xy \in \mathrm{Tod}\, e$. Now let us denote $\mathrm{Tod}\, e$ by T. By (iii), we have $N \subset T$ and $0 \in T$. Let $\mathscr{D}$ be the component of zero in T. Then $0 \in \mathscr{D} \subset T$ which is a maximal connected set contained in T. Since N is connected, we have $\{0\} \in N \subseteq \mathscr{D}$. For any element $y \in T$, $\mathscr{D}y$, being the continuous image of a connected set, is connected and contains zero. By the maximality of the set $\mathscr{D}$, we have $\mathscr{D}y \subseteq \mathscr{D}$. So $\mathscr{D}$ is a connected ideal of T and every element of N is contained in $\mathscr{D}$. Our proof is completed.

1.6 Counter-examples. Example 1. If S is not abelian, then N is not necessarily a subsemigroup of S. For example, let

$$S = \left\{0 = \begin{pmatrix}00\\00\end{pmatrix}, x = \begin{pmatrix}01\\00\end{pmatrix}, y = \begin{pmatrix}00\\10\end{pmatrix}, c = \begin{pmatrix}10\\00\end{pmatrix}, d = \begin{pmatrix}10\\01\end{pmatrix}\right\}.$$

Then under the ordinary matrix multiplication, we have the following multiplication table:

$\cdot$	0	x	y	c	d
0	0	0	0	0	0
x	0	0	c	0	x
y	0	d	0	y	0
c	0	x	0	c	0
d	0	0	y	0	d

Clearly, $N = \{0, x, y\}$, $N^2 = \{0, c, d\}$. Thus $N^2 \not\subset N$. This example demonstrates that the condition *abelian* is necessary.

Example 2. Even if S is an abelian semigroup, N is not necessarily an idempotent set. For let S be the real line with the usual topology. Define $x * y = 0$ for all x, y in S. Then $N = S$ but $N^2 = 0$.

Example 3. Even if N is connected, $\mathrm{Tod}\, e$ is not necessarily connected. For let

$$S = \{-1\} \cup \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{1\}.$$

Define $x*y = xy$ for x and $y \geqslant 0$:

$$x*y = -xy \quad \text{for } x \text{ and } y \leqslant 0,$$

$$x*y = 0 \quad \text{for } x > 0, y \leqslant 0 \text{ or } y \geqslant 0, x < 0;$$

then $\operatorname{Tod}\{1\} = \{-1\} \cup [-\frac{1}{2}, \frac{1}{2}]$, which is not connected.

Example 4. Even if S is connected, N is not necessarily connected. For let S be the two line segments joining the points $(0, 0)$, $(1, 0)$ and $(1, 0)$, $(0, 1)$. The topology of S is the usual topology inherited from the plane. Define the multiplication on S by $(x_1, y_1)*(x_2, y_2) = (\min(x_1, x_2), 0)$. Then S is clearly a connected abelian semigroup with zero. But $N = \{(0, 0), (0, 1)\}$ which is disconnected.

Example 5. The following example shows that there exists a k-ideal of S with $k > 1$.

Let $S = \{0, 1, 2, 3, 4, 5\}$ be a semigroup with the following multiplication table:

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	4	0	2
2	0	0	0	0	0	0
3	0	5	0	0	2	0
4	0	2	0	0	0	0
5	0	0	0	2	0	0

Let $A = \{0, 1, 2, 3, 4\}$; then $A^2 = \{0, 2, 4, 5\}$. Clearly, $AS \not\subset A$, but $A^2S = \{0, 2, 4\} \subset A$.

PROPOSITION 1.7. *If N' consists only of idempotent elements, then N is the topological radical of S. If N is the topological radical of S and G is any non-zero subgroup of S, then $G \subseteq N'$.*

Proof. To prove that N is the topological radical of S, it suffices to show that N is an ideal of S. But this is trivial since N' consists only of idempotent elements. If N is the topological radical of S, then N, in particular, is an ideal of S. Now let us suppose that $N \cap G \neq \emptyset$. Then there exists $x \in N \cap G$. As G is a subgroup of S, $Gx = xG = G$. Hence $G = Gx \subset SN \subset N$ which implies that $G = \{0\}$. Our supposition is therefore impossible.

PROPOSITION 1.8. *Let a be any arbitrary element in S. If N is an open subset of S, then* $\operatorname{Tod} a$ *is open. If N is a closed subset of S, then* $\operatorname{Tod} a$ *is closed.*

Proof. For each $x \in \operatorname{Tod} a$, we have $ax \in N$. According to our assumption, N is open and hence there exists a neighborhood $V(ax)$ of ax such that $V(ax) \subset N$. From the continuity of multiplication, there exist neighborhoods $V_1(a)$, $V_2(x)$ of the points a, x, respectively, such that $V_1(a)V_2(x) \subset V(ax) \subset N$. Hence $aV_2(x) \subset V_1(a)V_2(x) \subset N$. This means that $x \in V_2(x) \subseteq \mathrm{Tod}\,a$. Hence $\mathrm{Tod}\,a$ is an open subset of S. To prove that $\mathrm{Tod}\,a$ is closed if N is closed, it suffices to prove that $\overline{\mathrm{Tod}\,a} \subseteq \mathrm{Tod}\,a$. Let $x \in \overline{\mathrm{Tod}\,a}$. Then, since S is Hausdorff, there exists sequence $\{x_i\} \in \mathrm{Tod}\,a$ such that $x_i \to x$. This implies that $ax_i \in N$ for each i and, by the continuity of multiplication, we have $ax_i \to ax$. Since N is closed, we have $ax \in N$. This means that $x \in \mathrm{Tod}\,a$.

Remark 1.9. The following example shows that if N is not open, then not all $\mathrm{Tod}\,a$ are open sets.

Let $S = \{Z^-\} \cup [0, \infty)$, when $\{Z^-\}$ are the negative integers. The topology of S is the usual topology inherited from the real line. Define the multiplication $*$ in S by

$$x*y = \begin{cases} \min(x, y) & \text{whenever } x, y \in [0, \infty), \\ 0 & \text{if } x \in \{Z^-\}, y \in [0, \infty) \text{ and vice versa}, \\ -xy & \text{if both } x, y \in \{Z^-\}, \text{ where } xy \text{ is the ordinary} \\ & \text{multiplication.} \end{cases}$$

Clearly, $N = \{0\}$ is not open. For any $a \in [0, \infty)$, $\mathrm{Tod}\,a = \{0\} \cup \{Z^-\}$, which is open subset of S. For any $a \in \{Z^-\}$, $\mathrm{Tod}\,a = [0, \infty)$ which is not open.

Now let us call a semigroup S an *A-semigroup* if $\mathrm{Tod}\,a$ is an open subset of S for every $a \in S$. From Proposition 1.8 we know that if S is an N-semigroup, then S is an A-semigroup. But the converse statement is not known to the authors. That is, if S is an A-semigroup, is S an N-semigroup? (**P 796**)

The following gives a necessary and sufficient condition for the set N to be a k-ideal of S.

THEOREM 1.10. *Let S be connected and let $J_k(N)$ be the k-ideal generated by N. Then N is a k-ideal of S if and only if the component $\mathscr{C}$ of $\{0\}$ in N coincides with the component $\mathscr{D}$ of $\{0\}$ in $J_k(N)$.*

To prove this theorem, we show something more general, namely, if we replace the set N by any subsemigroup A of S containing zero, we shall see that our statement still holds.

Proof. Since A is a subsemigroup of S, we have $J_k(A) = A \cup A^2 \cup \cup \ldots \cup A^k S = A \cup A^k S$. Suppose A is a k-ideal of S. Then $A^k S \subseteq A$ and we have $J_k(A) = A$. Thus, the component $\mathscr{C}$ of $\{0\}$ in A and the component $\mathscr{D}$ of $\{0\}$ in $J_k(A)$ coincide. For the converse part, we first observe that $0 \in AS$. Also, for any $a \in A$, $(a, 0) \in (\{a\} \times S) \cap (S \times \{0\})$. Since S is connected,

$$\Big(\bigcup_{a \in A} (\{a\} \times S)\Big) \cup (S \times \{0\}) = (A \times S) \cup (S \times \{0\})$$

is a connected subset of $S \times S$. By the continuity of multiplication, we infer that AS is a connected subset of S containing 0. As a consequence, $0 \in A^k S$ and $A^k S$ is connected. Moreover, $J_k(A) \supseteq A^k S$. Thus the component $\mathscr{D}$ of $\{0\}$ in $J_k(A)$ contains $A^k S$. On the other hand, $A \subset J_k(A)$ and $\{0\} \in A$. Thus the component $\mathscr{C}$ of A is also in $J_k(A)$. By our assumption, $\mathscr{D} = \mathscr{C}$. Thus $A^k S \subset \mathscr{D} = \mathscr{C} \subset A$, that is, A is a k-ideal of S.

2. Abelian N-semigroups. We are now going to study, in this section, the structure of *N-semigroups*, that is, semigroups S in which the set N is an open subset of S. We use E to represent the collection of all idempotents of S and $E^* = E - \{0\}$. It is easily seen [6] that E is closed. An idempotent e may be looked upon as a subgroup of S. Following the usage of [2] and elsewhere, $H(e)$ is the maximal group containing an idempotent e. Also, by the notation of Koch [4] and Numakura [6], we define, for each x in S,

$$\Gamma_n(x) = \overline{\{x^i\}_{i=n}^{\infty}}, \qquad \Gamma(x) = \Gamma_1(x), \qquad K(x) = \bigcap \{\Gamma_n(x) \mid n \geqslant 1\}$$

and in case $\Gamma(x)$ is compact, we have $\Gamma(x) = \{x, x^2, \ldots\} \cup K(x)$, where $K(x)$ is the minimal ideal of $\Gamma(x)$ and is the maximal subgroup of $\Gamma(x)$. If $\Gamma(x)$ is compact for each element x in S, then S is called *elementwise compact*.

THEOREM 2.1. *If S is a compact N-semigroup (not necessarily abelian) which is not nil, then there always exists a compact subgroup of S which is disjoint from N.*

Proof. Since S is not nil, there exists at least an element $x \in S$ with $x \notin N$. By lemma 2.1.4 of [7], p. 58, $x \notin N$ implies $x^n \notin N$ for all positive integers n. Thus the sequence $\{x^n\}_{n=1}^{\infty} \subseteq N'$. Since N is open, N' is closed and hence compact. Therefore, $\Gamma(x) \subseteq N'$. Clearly, $\Gamma(x)$ is compact. So there exists a unique idempotent $0 \neq e^2 = e \in \Gamma(x)$. Consider $H(e)$ in $\Gamma(x)$. Since S is compact, $H(e)$ is a non-zero compact subgroup of S. We claim that $H(e) \cap N = \emptyset$. For otherwise, there exists an element y such that $y \in H(e)$ and $y \in N$. Since $H(e)$ is a group, we have $\{y^n\}_{n=1}^{\infty} \subseteq H(e)$. By the compactness of $H(e)$, there is a unique idempotent $e_1^2 = e_1 \in \Gamma(y) \subseteq H(e)$. On the other hand, $\{y^n\}_{n=1}^{\infty} \subseteq N$ and the set of cluster points of this sequence is the set $\{0\}$. We thus obtain that $e_1 = 0$. But if $0 \in H(e)$, then $H(e) = \{0\}$, which is false.

One would naturally ask: (**P 797**) Under what conditions can S be uniquely decomposed into two disjoint sets N and G, where N is the set of nilpotent elements and G is a compact subgroup of S, that is, we need $S = N \cup G$, $N \cap G = \emptyset$. The authors are unable to answer this question, however, algebraically, we can construct the following

THEOREM 2.2. *Let N be an abelian semigroup, G an abelian group which is disjoint from N. Define the multiplication $\odot$ in the set $S = G \cup N$ as follows:*

(a) *For $x, y \in G$, let $x \odot y = x * y$, where $*$ is the group multiplication.*

(b) *For $x, y \in N$, let $x \odot y = x \cdot y$, where $\cdot$ is the semigroup multiplication.*

(c) *For $x \in G$, $y \in N$, let $x \odot y = y = y \odot x$.*

Then S is an abelian semigroup, denoted by $S(N, G; \odot)$, in which N is the unique maximal proper ideal of S. In other words, N can be embedded as the unique maximal proper ideal in $S(N, G; \odot)$.

Proof. One can easily verify that the multiplication $\odot$ of S is associative, commutative and closed, hence $S(N, G; \odot)$ is an abelian semigroup. Also, it is easily seen that N is an ideal of S. We only have to show that N is the unique maximal proper ideal of S. In fact, suppose A is an ideal such that $A \not\subset N$. Then there exists $a \in A$, $a \notin N$. Thus, for each $x \in N$, we have $a \odot x = x \in AN \subseteq A$, that is, $N \subset A$. Suppose $N \subsetneqq A \subset S$. Then $G \cap A \neq \emptyset$. Let $b \in G \cap A$. As G is a group, $Gb = G \subseteq SA \subseteq A$. Hence $S = G \cup A \subseteq A$, which implies $S = A$. Thus $A \not\subset N \Rightarrow N = A$ or $A = S$, and N is indeed the unique maximal ideal of S.

Now, we give a characterization of abelian N-semigroups.

THEOREM 2.3. *Let S be an abelian compact semigroup which is not nil. Then S is an N-semigroup if and only if E^* is compact and non-empty.*

In order to prove this theorem, the following lemma is crucial:

LEMMA 2.4. *If S is an elementwise compact (or sequentially compact) abelian semigroup, then N is the topological radical of S.*

Proof. (i) Suppose S is elementwise compact. We wish to show that N is an ideal of S. Let $x \in N$ and $y \in S$. Consider $\Gamma(y)$ which, by our assumption, is compact. Take any $z \in \Gamma(y)$. We have $z0 = 0$. Thus, by the continuity of multiplication, for any arbitrary neighborhood U of 0, there exist neighborhoods $w(z) \in \mathscr{G}(z)$, $w_z(0) \in \mathscr{G}(0)$ such that $w(z)w_z(0) \subseteq U$, where $\mathscr{G}(z)$ and $\mathscr{G}(0)$ are complete systems of neighborhoods of the elements z and 0, respectively. Let us consider a system of neighborhoods $\{w(z) \mid z \in \Gamma(y)\}$. It is evident that

$$\Gamma(y) \subset \bigcup_{z \in \Gamma(y)} w(z).$$

Since $\Gamma(y)$ is a compact semigroup, there exists a finite system $w(z_1)$, $w(z_2), \ldots, w(z_n)$ which also covers $\Gamma(y)$ and, for $i = 1, 2, \ldots, n$, we have $w(z_i)w_{z_i}(0) \subseteq U$. Evidently, there exists a neighborhood $w(0) \in \mathscr{G}(0)$ such that

$$w(0) \subset \bigcup_{i=1}^{n} w(z_i) \quad \text{and} \quad w(z_i) \cdot w(0) \subseteq U$$

for every $i = 1, 2, \ldots, n$. But $w(0)$ is a neighborhood of 0 and $x \in N$, and hence $x^n \in w(0)$ for $n \geqslant n_0$ for some n_0. Thus, for $n \geqslant n_0$,

$$y^n x^n \in \Gamma(y)\cdot w(0) \subseteq \bigcup \{w(z_i) \mid i = 1, 2, \ldots, n\}\ w(0) \subseteq \bigcup U = U.$$

This means that $xy \in N$, that is, N is an ideal of S.

(ii) Now suppose S is sequentially compact and N is not an ideal of S. Then we can find $x \in N$ and $y \in S$ such that $xy \notin N$. That is, $(xy)^n \not\to 0$. Thus

(*) for any open neighborhood V of $\{0\}$, there exists a subsequence $(xy)^{n_k}$ of $(xy)^n$ such that $(xy)^{n_k} \notin V$ for every $k = 1, 2, \ldots$

Consider the subsequence $\{y^{n_k} \mid k = 1, 2, \ldots\}$. Since S is sequentially compact and Hausdorff, there exists a subsubsequence $\{y^{n_{k_i}} \mid i = 1, 2, \ldots\}$ of $\{y^{n_k}\}$ such that $y^{n_{k_i}} \to s \in S$.

Clearly,

$$\lim_i (xy)^{n_{k_i}} = \lim_i (x^{n_{k_i}} y^{n_{k_i}}) = \lim_i x^{n_{k_i}} \lim_i y^{n_{k_i}} \to 0\cdot S = 0.$$

This, however, contradicts (*). So, indeed, N is an ideal of S.

COROLLARY. *If S is a compact abelian semigroup with zero, then N is an ideal of S.*

We are now ready to prove Theorem 2.3. Suppose that S is an abelian compact N-semigroup. Then N is an open subset of S. Clearly, $E^* = N' \cap E$. As N is open, N' is closed. Also, it is well known [6] that E is closed, so by the compactness of S, N' and E are compact subsets of S. Therefore E^* is compact and non-empty. Conversely, let us suppose E^* is compact and non-empty. Then $S - E^*$ is open and $N \subset S - E^*$. Clearly, $J_0(S - E^*)$, the union of all ideals of S contained in $S - E^*$, is open [7]. By our Lemma 2.4, N is an ideal of S and hence $N \subseteq J_0(S - E^*)$. Suppose, if possible, $N \subsetneqq J_0(S - E^*)$. Then there exists $x \in J_0(S - E^*)$, $x \notin N$. Consider the principal ideal $J(x)$ generated by x. Then $J(x) = x \cup xS \subseteq J_0(S - E^*)$. Since S is compact, so is $J(x)$. Hence $\Gamma(x) \subseteq J(x)$ and there exists a unique idempotent element $e^2 = e \in \Gamma(e) \subseteq J_0(S - E^*)$ (cf. [7]). Clearly, $e = 0$, for otherwise $e \notin J_0(S - E^*)$. But if $e = 0$, then $K(x) = e\Gamma(x) = \{0\}$. Since $K(x)$ is the set of cluster points of the sequence $\{x^n\}_{n=1}^{\infty}$, we would have $x^n \to 0$. This contradicts $x \notin N$. Thus we conclude that $N = J_0(S - E^*)$, which is an open subset of S.

COROLLARY 1. *Let S be compact. If there exists an idempotent $e \neq 0$ such that $N \neq \mathrm{Tod}\, e$, then $\mathrm{Tod}\, e$ contains at least one non-zero idempotent of S.*

Proof. By Lemma 2.4, N is an ideal of S. Since $N \neq \mathrm{Tod}\, e$, so by Proposition 1.5 (iii) and (iv), $N \subsetneqq \mathrm{Tod}\, e$ and $\mathrm{Tod}\, e$ is an ideal of S. Applying the same argument as in the proof of Theorem 2.3, we see that there exists $0 \neq f^2 = f \in \Gamma(x) \subset \mathrm{Tod}\, e$.

COROLLARY 2. *If S is compact and D is the component of* 0 *in N, then $NS \subseteq D$.*

COROLLARY 3. *Let S be compact. If $E \subseteq \bar{N}$ and $\bar{N} \neq S$, then $\bar{N}$ is contained in a proper compact ideal of S.*

Proof. Suppose $\bar{N} \neq S$. According to our Lemma 2.4, $\bar{N}$ is a proper ideal of S. Thus $\bar{N}$ is contained in all proper maximal ideals M_i of S, where M_i has the form $J_0(S-x_i)$ for some $x_i \in S-\bar{N}$. Hence $E \subset \bar{N} \subset M_i$ for all i. By Koch and Wallace [7], p. 44, we have $S^2 \subset M_i$ for all i, that is, $S^2 \subset \bigcap_i M_i$. But since S is compact, by Koch and Wallace again [5], $\bigcap_i M_i \subset S^2$. Hence $S^2 = \bigcap_i M_i$. Clearly, $\bigcap_i M_i$ is compact and $\bar{N} \subset \bigcap_i M_i$.

THEOREM 2.5. *If S is a compact Ω-semigroup such that $\bar{N}^2 \not\subset N$, then $\bar{N}-N$ is either a group or a semilattice of groups.*

To prove Theorem 2.5, we need the following

LEMMA 2.6. *Let S be a compact Ω-semigroup. If I is a minimal ideal containing N properly, then $I-N$ is a closed non-nil subsemigroup of S if and only if $I^2 \not\subset N$.*

Proof. One part is trivial. For the other part, let us consider $I-N$. We claim that $I-N$ is a subsemigroup of S. For let $a, b \in I-N$. Then $a^n, b^n \in I-N$ for every $n = 1, 2, \ldots$ Now, let $ab \in N$. Consider $J(a) = a \cup Sa \subset I$, which is the principal ideal generated by a. As $a \notin N$ and by Lemma 2.4, N is an ideal of S, we have $J(a) \not\subset N$. Since S is an Ω-semigroup, we obtain $N \subsetneqq J(a) \subset I$. However, by the minimality of I containing N, we conclude that $J(a) = I$. Similarly, $J(a) = J(a^n) = I$ for every $n = 1, 2, \ldots, J(b) = J(b^n) = I$ for every $n = 1, 2, \ldots$ Now $J(a)J(b) = (a \cup Sa)(b \cup Sb) = ab \cup Sab = J(ab)$. So $I^2 = J(ab) \subseteq I$. According to our assumption, $I^2 \not\subset N$. Therefore, we have $N \subsetneqq I^2 \subseteq I$, which implies $J(ab) = I$. Similarly, $J((ab)^n) = I$ for every $n = 1, 2, \ldots$ Hence $\bigcap_n J((ab)^n) = I$. Since $ab \in N \Rightarrow (ab)^n \to 0$, we infer that $I = \bigcap_n J((ab)^n) = \{0\}$ which is a contradiction, for we assumed that I contains N properly. This establishes our claim. Now take $a \in I-N$. Then $a, a^2, \ldots$ are all in $I-N$. Hence

$$\Gamma(a) = \overline{\{a^n\}_{n=1}^{\infty}} \subseteq J(a)$$

and $\Gamma(a)$ is compact. Thus there exists a unique idempotent $f = f^2 \in \Gamma(a) \subseteq I$. Since $a \notin N$, by the same argument as in Theorem 2.3, we see that $f \neq 0$. Hence $I-N$ contains a non-zero idempotent and $I-N$ is a closed non-nil subsemigroup of S.

We now prove Theorem 2.5. By Lemma 2.4, $\bar{N}$ is an ideal of S which contains N properly. We claim that $\bar{N}$ itself is the minimal ideal containing N properly. For suppose that there exists an ideal N_1 such that $N \subsetneqq N_1$

$\subset \bar{N}$. Take $x \in N_1 - N$. Then, as S is an Ω-semigroup, we have $N \subsetneqq J(x) = x \cup xS \subseteq N_1 \subseteq \bar{N}$. As S is compact, $J(x)$ is closed. Hence by the definition of the closure of N, $\bar{N} = J(x)$, which implies $N_1 = \bar{N}$. Now apply our Lemma 2.6; $\bar{N}$ is a minimal non-nil ideal of S. So there exists a non-zero idempotent $e^2 = e \in \bar{N} - N$ such that $J(e) = e \cup Se = \bar{N}$. This implies that $\bar{N} = eS$. Hence, by a result of Koch [7], p. 57, $eS - N$ is a group and e is primitive. Moreover, if $\bar{N}$ contains more than one idempotent, then $eS - N$ is the disjoint union of the maximal groups $e_a S - N$ for all $e_a \in \bar{N} - N$ (cf. [7], p. 61). In other words, $eS - N$ is either a group or a semilattice of groups.

The following is a slight modification of a theorem given by Numakura [6], p. 407, Theorem 4. However, for the sake of completeness, we give the proof in detail.

THEOREM 2.7. *If S is locally compact and N is a compact ideal of S, then for any open neighborhood V of N, there always exists an open non-nil subsemigroup J of S such that $N \subset J \subset V$.*

Proof. Since S is locally compact and Hausdorff, S is regular and we can find a neighborhood U of N having compact closure such that $N \subset U \subset \bar{U} \subset V$, where V is any open neighborhood containing N. Since N is an ideal, $N\bar{U} \subset N \subset U$. By the continuity of the multiplication and compactness of $\bar{N}$ and $\bar{U}$, we can find an open set W with $N \subset W \subset U$, and $W\bar{U} \subset U$. Since $W \subset \bar{U}$, $W^2 \subset W\bar{U} \subset U$. Similarly, $W^3 \subset U, \ldots$ and hence $\bigcup_n W^n \subset U$. Write $T = \overline{\bigcup_n W^n}$. T is clearly a compact subsemigroup of S contained in V. Now let $J = J_0(W)$, the union of all ideals contained in W. Therefore $J \subset W \subset T \subset V$. Since S is compact, J is therefore open and is a subsemigroup of S. Since N is an ideal contained in W, $N \subsetneqq J \subset W \subset V$. Clearly, J is non-nil, for otherwise, we would have $J \subseteq N$, which is false.

3. Semigroups with zero and local zeros. In this section, we shall consider abelian semigroups with zero and local zeros. Throughout, e^* will stand for a local zero in S.

THEOREM 3.1. *The closure of N contains no local zeros.*

Proof. Let $\bar{N} = T$ and suppose T has a local zero. As N is dense in T, then for any $t \in T$ and any neighborhood $V(t)$ of t we have $V(t) \cap N \neq \emptyset$. In particular, if e^* is a local zero of T, $V(e^*) \cap N \neq \emptyset$, where $V(e^*)$ is such that $x \in V(e^*)$ implies $xe^* = e^*x = e^*$. Let us take $x \in V(e^*) \cap N$; then $e^*x = xe^* = e^*$ and, by definition of local zero, $e^* \notin N$. However, since $x \in N$, we would have $(e^*x)^k = e^*x^k \to 0$. This means that $e^* = 0$, which is a contradiction. We therefore conclude that $\bar{N}$ has no local zeros.

The next theorem tells us that if S contains zero and local zeros, then S must be disconnected.

THEOREM 3.2. *If P is a connected subset of S which has a local zero $e^* \in P$, then $\bar{P} \cap \bar{N} = \emptyset$.*

Proof. We first claim that e^* is the zero element for the subset $\bar{P}$. Now $\bar{P}$ is connected. Hence $e^* \bar{P}$ is a connected subset of $\bar{P}$. Clearly, $e^* \in e^* \bar{P}$. As e^* is a local zero for S, then there is an open neighborhood U in S such that $e^* \in U \cap \bar{P} = V$, $e^* x = x e^* = e^*$ for all $x \in V$. Hence $e^* \bar{P} \cap V \neq \emptyset$. V is open in $\bar{P}$. Thus $e^* \bar{P} \cap V$ is open in $\bar{P}$. Consider $x \in e^* \bar{P} \cap V$. Now $x \in e^* \bar{P}$ and $x \in V$ imply that $x = e^* p$, $x \in V$ with $p \in \bar{P}$, and this implies that $x = e^{*2} p = (e^* p) e^* = x e^* = e^*$. Hence $\{e^*\} = e^* \bar{P} \cap V$ is open in $\bar{P}$. But $\bar{P}$, being the subspace of a Hausdorff space, is Hausdorff. Hence $\{e^*\}$ is closed in $\bar{P}$ as well. Thus $\{e^*\}$, being an open and closed subset of the connected set $e^* \bar{P}$, must be equal to $e^* \bar{P}$. That is, $e^* \bar{P} = \{e^*\} = \bar{P} e^*$. Our claim is established. Now let us suppose that $\bar{P} \cap \bar{N} \neq \emptyset$. Then there exists $x \in \bar{P} \cap \bar{N}$. Hence $x \in \bar{P}$ and $x \in \bar{N}$. Since e^* is the zero for $\bar{P}$, $x e^* = e^*$. By the continuity of the multiplication, for any arbitrary neighborhood U of e^*, there exists a neighborhood V of x such that $V e^* \subset U$. But since $x \in \bar{N}$, $V \cap N \neq \emptyset$. Let $y \in V \cap N$; then $y \in N$ and $y e^* \subset U$. As U is a neighborhood of e^*, this implies that $(y e^*) e^* = e^* (y e^*) = e^*$. That is $y e^* = e^*$. But then, $(e^*)^n = (y e^*)^n = y^n e^* \to 0$, which is a contradiction. So $\bar{P} \cap \bar{N} = \emptyset$.

COROLLARY 1. *If N' is a connected subset of S containing e^*, then* $\operatorname{Tod} e^*$ *is a prime ideal containing N.*

Proof. N is an ideal of S since S is compact, by Lemma 2.4. By Proposition 1.5 (iii) and (iv), $N \subset \operatorname{Tod} e^*$ and $\operatorname{Tod} e^*$ is an ideal of S. We only need to verify that $\operatorname{Tod} e^*$ is prime. For this purpose, let us consider $ab \in \operatorname{Tod} e^*$, $a \notin \operatorname{Tod} e^*$. Then it follows that $abe^* \in N$ and $ae^* \notin N$. Since $ae^* \notin N$, we have $ae^* \in N'$. Now, by our theorem, e^* acts as a zero in N'. Hence $(ae^*) e^* = e^*$, that is, $ae^* = e^*$. Thus $abe^* \in N \Rightarrow b(ae^*) \in N \Rightarrow be^* \in N \Rightarrow b \in \operatorname{Tod} e^*$. Thus $\operatorname{Tod} e^*$ is indeed a prime ideal of S containing N.

COROLLARY 2. *If e is any non-zero idempotent of S such that $e^* \notin$* $\operatorname{Tod} e$, *then* $\operatorname{Tod} e \cap \mathscr{C} = \emptyset$, *where $\mathscr{C}$ is the component of e^* in S.*

The following theorem concerns the existence of a local zero in abelian semigroups with zero.

THEOREM 3.3. *Let S be a compact abelian N-semigroup. If $N' = E^*$, which is a connected subset of S and is disjoint from N, then S contains a local zero. Furthermore, if S has a unit and N' is arcwise connected, then N' is contractible.*

The proof of this theorem is a consequence of the following

THEOREM 3.4. *Let S be a compact connected semigroup (not necessarily abelian) such that $S = ES = SE$ and E is an abelian submob of S. Then S has an idempotent e such that $eE = Ee = e$ and the minimal ideal $M(S) = H(e) = eSe = eS = Se$. Moreover, $H^*(S)$ is isomorphic to $H^*(eSe)$.*

Proof. Let $M(S)$ be the minimal ideal of S. Then, by 1.22 of [2], we can find a primitive idempotent e in $M(S)$ such that eSe is a group and $eSe = eM(S)e$. Let $H(e)$ be the maximal subgroup of S containing e. Then one can easily verify that $H(e) = eSe = eM(S)e$. By Theorem 1.2.11 of [7], p. 34, we therefore have $M(S) = SeS$, which is a two-sided ideal of S. Hence, by Lemma 1.2.8 of [7], $M(S) = (Se \cap E)eSe(eS \cap E)$. We now show that under the condition of the theorem we have $Se \cap E = eS \cap E = \{e\}$.

Suppose $e_1 \in Se \cap E$. Then we can write $e_1 = xe$ for some $x \in S$. Hence $ee_1 = exe \in eSe = H(e)$. Since E is abelian, ee_1 is an idempotent and hence $ee_1 = e$. Thus $e_1 = xe = (xe)e = e_1e = e$. Similarly, we have $eS \cap E = \{e\}$. Thus we have established that $M(S) = eSe = H(e)$. Since $M(S)$ is closed, we have $H^*(S) \cong H^*(eSe)$. It remains for us to show that $eE = Ee = \{e\}$. Let $e_1 \in E$. Since $H(e) = M(S)$ is an ideal, we have $H(e)S \subset H(e)$, $SH(e) \subset H(e)$. Thus $ee_1 \in H(e)$, $e_1e \in H(e)$. Since ee_1 and e_1e are idempotents, we have $ee_1 = e_1e = e$. It only remains to show that $M(S) = Se = eS$. In fact, let x be an element of $M(S)$. Then $x \in H(e)$; we have $x = ex \in Sx$. Thus $H(e) = M(S) \subset Sx$. On the other hand, $Sx \subset SM(S) \subset M(S)$. Thus $H(e) = M(S) = Sx$. Similarly, $M(S) = xS$ for any $x \in M(S)$. In fact, for any $x \in M(S)$, it is easily seen that we have $xM(S) = M(S)x = M(S) = Sx = xS$.

We now prove Theorem 3.3.

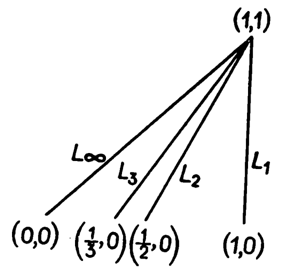
Since S is a compact abelian N-semigroup, by Theorem 2.3, we have $N' = E^*$ is compact and non-empty. Therefore E^* is a compact subsemigroup of S. The hypotheses of Theorem 3.4 on N' are satisfied. Hence, by Theorem 3.4, we can find an idempotent e^* such that $e^*N' = N'e^* = \{e^*\}$, that is, S has a local zero e^*, and e^* is the zero of N'. Moreover, by Theorem 3.4, N' is acyclic. Now suppose S has a unit u. Then clearly, $u \in E^*$. As N' has a unit and is arcwise connected, we apply a result of Gottlieb and Rothman [1]. Recall that they say that the semigroup N' satisfies * if for each x in N', there is an element y such that $xy = y$. Since N' has a zero, we see that N' satisfies *. Then by Lemma 1 of [1], we infer that N' is contractible.

THEOREM 3.5. *Let S have a zero and local zero e^*. If $N' = E^*$ is connected, then* $\operatorname{Tod} e^* = \bigcup_{a \in E^*} \operatorname{Tod} a$.

Proof. Since $N' = E^*$, by Proposition 1.7, N is an ideal of S. By Proposition 1.5 (iv), $\operatorname{Tod} x$ is an ideal of S for every $x \in S$. Suppose if possible, $\operatorname{Tod} a \subsetneqq \operatorname{Tod} e^*$ for all $a \in N'$; then there exists $x \in \operatorname{Tod} e^*$, $x \notin \operatorname{Tod} a$. As $\operatorname{Tod} e^*$ is an ideal of S, we have $xa \in \operatorname{Tod} e^*$, which implies $axe^* \in N$. But since $x \notin \operatorname{Tod} a$, we have $ax \notin N$. Hence $ax \in N'$ which is connected and is the set of non-zero idempotents of S. So, by Theorem 3.2, e^* is the zero for N'. Thus $axe^* = e^* \in N'$, a contradiction.

4. An example. In this section, we construct an example to show that even if S is not compact, but S is locally compact and not locally connected, some important properties concerning the set N, which we have just discussed in sections 2 and 3, are still valid.

Example. Let S^* be the subset of the plane E^2 consisting of line segments L_n, joining the points $(1, 1)$ and $(1/n, 0)$ for all $n = 1, 2, \ldots$ Let $S = S^* - (1, 1)$. The topology of S is the usual topology inherited from E^2.



For any point (x_n, y_n) on L_n, we have

$$x_n = \lambda_n \frac{1}{n} + (1-\lambda_n) = \lambda_n\left(\frac{1}{n} - 1\right) + 1,$$

$$y_n = (1-\lambda_n),$$

where $0 \leqslant \lambda_n < 1$.

Define the multiplication $\hat{*}$ on S as follows:

$$(x_n, y_n) \mathbin{\hat{*}} (x_m, y_m) = \left(\lambda_n\left(\frac{1}{n} - 1\right) + 1, 1 - \lambda_n\right) \mathbin{\hat{*}} \left(\lambda_m\left(\frac{1}{m} - 1\right) + 1, 1 - \lambda_m\right)$$

$$= \left(\delta\left(\frac{1}{mn} - 1\right) + 1, \sigma\right),$$

where $\delta = \min(\lambda_n, \lambda_m)$ and $\sigma = \min(1-\lambda_m, 1-\lambda_n) = \min(y_n, y_m)$. Clearly, $\hat{*}$ is closed and associative and S is a semigroup.

To see that S is a topological semigroup, we have to verify that $*$ is a continuous mapping from $S \times S$ into S. It suffices for us to check that $\hat{*}$ is continuous at $(0, 0)$, for the continuity at other points is clear. It is easy to check continuity at $(0, 0)$.

Then S is a topological semigroup with zero.

In this example, S is locally compact but not locally connected. The point $(1, 0)$ is a local zero for S.

$$\bar{N} = (0, 0) \cup \left\{\left(\frac{1}{n}, 0\right) \middle| n = 2, 3, \ldots\right\},$$

which is totally disconnected and has no local zeros of S. $\bar{N}$ is an ideal of S. The component of $(1, 0)$ in S is clearly disjoint from $\bar{N}$. The topological zero divisors of $e^* = (1, 0)$ is the set $S - L_1$ which is the maximal ideal among the ideals $\{\mathrm{Tod}\, a \mid a \in N'\}$. Moreover, $\mathrm{Tod}\, e^*$ is a prime ideal containing N. Since $N' \neq E^*$, we can see that S is not an N-semigroup.

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