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SIDON SETS AND I_0-SETS

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In this note, $M(T)$ denotes the space of bounded Borel measures on the circle T with the total variation norm and $\hat{\mu}$ denotes the Fourier–Stieltjes transform of $\mu \in M(T)$. Let E be a subset of the integers Z and let φ be a bounded complex-valued function on E. The set E is called a *Sidon set* if to each φ there corresponds a $\mu \in M(T)$ such that $\varphi(m) = \hat{\mu}(m)$ for all $m \in E$. The set E is called an I_0-*set* (see [2] and [3]) if we may take the measure μ to be discrete. Clearly, every I_0-set is a Sidon set, and it is easy to exhibit Sidon sets which are not I_0-sets [1]. However, it is an open problem whether or not every Sidon set is a finite union of I_0-sets.

The purpose of this note is to shed some light on the difficulty in distinguishing Sidon sets from I_0-sets. We prove a theorem that shows discrete measures are just as efficient in a Fourier analytic sense as arbitrary bounded Borel measures in interpolating bounded functions on finite subsets of Z. Thus one cannot determine whether a Sidon set E is a finite union of I_0- sets merely by examining the norms of interpolating discrete measures on finite subsets of E.

THEOREM. *Let $K > 1$. To any $\mu \in M(T)$ and any finite set $F \subset Z$ there corresponds a discrete $\sigma \in M(T)$ satisfying*

(i) $\hat{\sigma}(m) = \hat{\mu}(m)$ *for all* $m \in F$,

(ii) $\|\sigma\| \leqslant K\|\mu\|$.

Proof. Choose $\varepsilon > 0$ such that $(1+\varepsilon)^2 \leqslant K$. Let P be any trigonometric polynomial such that $\hat{P} = 1$ on F and $\|P\|_1 \leqslant 1+\varepsilon$. Then $Q = P * \mu$ is a trigonometric polynomial satisfying

$$(1) \qquad \hat{Q}(m) = \hat{\mu}(m) \quad \text{for all } m \in F$$

and

$$(2) \qquad \|Q\|_1 \leqslant (1+\varepsilon)\|\mu\|.$$

For each $n \in \boldsymbol{Z}^+$ define a discrete measure on $\boldsymbol{T}$ by

$$\tau_n = (1/n) \sum_{k=1}^{n} \delta(2\pi k/n).$$

(Here $\delta(t)$ denotes the unit point mass at t.) If we form the discrete measure $\sigma_n = \tau_n \cdot Q$, then $\hat{\sigma}_n = \hat{\tau}_n * \hat{Q}$. If n is sufficiently large, it follows from the definition of the Riemann integral that

$$\|\sigma_n\| \leqslant (1+\varepsilon)\|Q\|_1. \tag{3}$$

Observe that $\hat{\tau}_n(k) = 1$ if $k = jn$ for some integer j and $\hat{\tau}_n(k) = 0$ otherwise. Thus

$$\hat{\sigma}_n(m) = \sum_{k \in \boldsymbol{Z}} \hat{\tau}_n(k) \hat{Q}(m-k) = \sum_{j \in \boldsymbol{Z}} \hat{Q}(m-jn). \tag{4}$$

If $n > |k|$ for all $k \in F - \operatorname{supp} \hat{Q}$, then $m \in F$ and $0 \neq j \in \boldsymbol{Z}$ imply $m-jn \notin \operatorname{supp} \hat{Q}$. With such a choice of n, (4) reduces to

$$\hat{\sigma}_n(m) = \hat{Q}(m) \quad \text{for all } m \in F. \tag{5}$$

Therefore, for sufficiently large n, $\sigma = \sigma_n$ satisfies (i) by (1) and (5), while (2), (3), and the inequality $(1+\varepsilon)^2 \leqslant K$ show that (ii) is also met.

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