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ON SOME CLASSES OF VOLTERRA INTEGRAL EQUATIONS IN BANACH SPACE

BY

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1. Introduction. In [6] McClure and Wong studied the existence and uniqueness of solutions to certain infinite systems of linear and nonlinear differential equations whose coefficients are not necessarily constants. Under certain conditions it is proved that the system of linear equations has a unique solution. This leads to a definition of an evolution operator for the linear system, which is very useful in the study of the nonlinear system.

In the present paper we are interested in some Volterra integral equations. These equations are significant generalizations of infinite systems considered in [6]. We establish modifications of the well-known fixed point principle, and use this result to solve our problems.

2. Fixed point theorems. Throughout this section, E will denote a Fréchet space and $\boldsymbol{P} = \{p_k\colon k = 1, 2, \ldots\}$ a family of seminorms which generates the topology of E (see, e.g., [7]).

Assume that f_0, f_i $(i = 1, 2, \ldots)$ are mappings of E into itself such that

$$\lim_{i\to\infty} f_i x = f_0 x \quad \text{for every } x \in E$$

and $p(f_i x - f_i y) \leqslant qp(x-y)$ for all $p \in \boldsymbol{P}$, $i \geqslant 1$, and $x, y \in E$, where q $(0 \leqslant q < 1)$ is a constant. From the Cain and Nashed theorem ([2], Theorem 2.2) we infer that f_i $(i = 1, 2, \ldots)$ and f_0 have unique fixed points x_i and x_0, respectively. Moreover,

$$p(x_i - f_i x_0) \leqslant (1-q)^{-1} qp(x_0 - f_i x_0) \quad \text{for all } p \in \boldsymbol{P} \text{ and } i \geqslant 1.$$

Hence

$$\begin{aligned} p(x_i - x_0) &\leqslant p(x_i - f_i x_0) + p(f_i x_0 - x_0) \\ &\leqslant (1-q)^{-1} qp(f_0 x_0 - f_i x_0) + p(f_i x_0 - f_0 x_0) \\ &= (1-q)^{-1} p(f_i x_0 - f_0 x_0) \end{aligned}$$

for all $p \in \boldsymbol{P}$ and $i \geqslant 1$. This implies that

$$\lim_{i\to\infty} x_i = x_0.$$

Now, applying the above remarks we obtain the following

PROPOSITION 1. *Let X be a nonempty subset of E. Let T, G_0, and G_n $(n = 1, 2, \ldots)$ be transformations defined on X with the values in E such that $T[X]$ is a closed set, $G_n[X] \subset T[X]$ for all $n \geqslant 1$,*

$$\lim_{n\to\infty} G_n x = G_0 x \quad \textit{for all } x \in X,$$

and $p(G_n x - G_n y) \leqslant qp(Tx - Ty)$ for all $p \in \boldsymbol{P}$, $n \geqslant 1$, and $x, y \in X$, where q $(0 \leqslant q < 1)$ is a constant.

Then for $j = 0, 1, \ldots$ there exists a unique point $y_j \in T[X]$ with the following properties:

(i) *$G_j x = Tx$ for every x such that $Tx = y_j$;*

(ii) *if $G_j x^{(i)} = Tx^{(i)}$ for $i = 1, 2$, then $Tx^{(1)} = Tx^{(2)}$;*

(iii) *$\lim_{n\to\infty} Tx_n = Tx_0$ for all x_n, x_0 such that $Tx_n = y_n$ and $Tx_0 = y_0$.*

Proof. Let $j = 0, 1, \ldots$ Fix $y \in T[X]$. Suppose that $v_i = G_j u_i$ $(i = 1, 2)$ with $Tu_i = y$. We have $p(v_1 - v_2) \leqslant qp(Tu_1 - Tu_2) = 0$ for every $p \in \boldsymbol{P}$. Since $\boldsymbol{P}$ is a saturated family of seminorms on E, we get $v_1 = v_2$. Consequently, $\{G_j x \colon Tx = y\}$ contains only one element.

By the application of our remarks we conclude that the mapping $y \mapsto \{G_j x \colon Tx = y\}$ of $T[X]$ into itself has a unique fixed point y_j and

$$\lim_{n\to\infty} y_n = y_0.$$

Further, if x_j is such that $Tx_j = y_j$, then

$$Tx_j = G_j x_j \quad \text{and} \quad \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y_0 = Tx_0.$$

Moreover, for $x^{(i)}$ $(i = 1, 2)$ with $G_j x^{(i)} = Tx^{(i)}$ we obtain the inequality $p(Tx^{(1)} - Tx^{(2)}) \leqslant qp(Tx^{(1)} - Tx^{(2)})$ for each $p \in \boldsymbol{P}$, and therefore $Tx^{(1)} = Tx^{(2)}$, which completes the proof.

PROPOSITION 2. *Let X, Y be nonempty subsets of E, and let Y be convex and closed. Suppose that T is a mapping from X to Y with $T[X]$ closed, and Q is a continuous mapping from Y into a compact subset of E. Further, assume that F is a mapping from $X \times Y$ into $T[X]$ satisfying the following conditions:*

(i) *$p\big(F(x_1, y) - F(x_2, y)\big) \leqslant qp(Tx_1 - Tx_2)$ for all $x_1, x_2 \in X$, $y \in Y$, and $p \in \boldsymbol{P}$, where q $(0 \leqslant q < 1)$ is a constant;*

(ii) *$p\big(F(x, y_1) - F(x, y_2)\big) \leqslant C_p p(Qy_1 - Qy_2)$ for all $x \in X$, $y_1, y_2 \in Y$, and $p \in \boldsymbol{P}$, where $C_p > 0$ is a constant (depending on the seminorm p).*

Then there exists a point $x_0 \in X$ such that $F(x_0, Tx_0) = Tx_0$.

Proof. Let us fix $y \in Y$. By Proposition 1 there exists a point $u_y \in X$ such that $F(u_y, y) = Tu_y$. Now, we consider the mapping $y \mapsto Tu_y$ of Y into itself. This operator is continuous and has values in a compact subset of E. For example, we prove that it is continuous on Y.

Let (y_n) be a convergent sequence in Y, and let

$$\lim_{n\to\infty} y_n = y_0.$$

Moreover, let us put $G_n x = F(x, y_n)$ and $G_0 x = F(x, y_0)$ for $x \in X$. Obviously, $G_n[X] \subset T[X]$ and $p(G_n u - G_n v) \leqslant qp(Tu - Tv)$ for $p \in P$ and $u, v \in X$. Since Q is continuous, by (ii) we obtain

$$\lim_{n\to\infty} G_n x = \lim_{n\to\infty} F(x, y_n) = F(x, y_0) = G_0 x \quad \text{for } x \in X.$$

Therefore, by Proposition 1,

$$\lim_{n\to\infty} Tu_{y_n} = Tu_{y_0}.$$

Finally, by the Singbal fixed point theorem (see [1], p. 169, or [2], Theorem 2.1 (b)) there exists $z \in Y$ such that $Tu_z = z$. Hence $Tu_z = F(u_z, z) = F(u_z, Tu_z)$, which completes the proof.

3. Statement of the problems. Assumptions (a)-(c), given below, are valid throughout this paper and will not be repeated in the formulations of particular theorems. Suppose that

(a) $I_s = [s, \infty)$, where s is a nonnegative real number, and

$$D_0 = \{(t, \tau) \in I_0 \times I_0 : 0 \leqslant \tau \leqslant t\};$$

(b) E is a Banach space with the norm $\|\cdot\|$, and $\mathscr{L}(E)$ is a Banach algebra of all linear continuous operators from E into itself with the norm $|\cdot|$;

(c) $f: I_0 \times E \to E$ is a continuous function.

Moreover, let us denote:

by $C(I_s, E)$ the set of all continuous functions defined on I_s with values in E;

by $\mathscr{U}$ the set of all mappings B from I_0 into $\mathscr{L}(E)$ such that $t \mapsto B(t)$ is a continuous operator-valued function (i.e., $t \mapsto B(t)x$ is a strongly continuous E-valued function for each $x \in E$);

by $\mathscr{V}$ the set of all mappings K from D_0 into $\mathscr{L}(E)$ such that $(t, \tau) \mapsto K(t, \tau)$ is a continuous operator-valued function (in the above operator sense) as a function of two variables and

$$|K(t, \tau)| \leqslant \exp\Big(\int_\tau^t \omega(\sigma)\,d\sigma\Big) \quad \text{for each } (t, \tau) \in D_0,$$

where ω is a real function (which does not depend on K) locally integrable on I_0.

We shall consider the integral equations

$$(+)\qquad y(t) = K(t, s)x + \int_s^t K(t, \tau)B(\tau)y(\tau)d\tau,$$

$$(++)\qquad y(t) = K(t, s)x + \int_s^t K(t, \tau)B(\tau)y(\tau)d\tau + \int_s^t K(t, \tau)f\big(\tau, y(\tau)\big)d\tau$$

with $s \geqslant 0$, $x \in E$, $B \in \mathscr{U}$, and $K \in \mathscr{V}$, where $y \in C(I_s, E)$ is an unknown function and all the integrals are the integrals in the strong sense.

In the sequel we shall use the notation

$$U(t) = \int_s^t \big(\omega(\tau) + q\|B(\tau)\|\big)d\tau \qquad \text{for } t \geqslant s,$$

where $q > 1$ is a constant, $B \in \mathscr{U}$, and ω is a function from the set $\mathscr{V}$.

4. Main results. The set $C(I_s, E)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of I_s. This topology is determined by the sequence (p_n) of seminorms defined by

$$p_n(y) = \sup_{s \leqslant t \leqslant n} \|y(t)\| \qquad \text{for } y \in C(I_s, E),$$

and therefore (see [7], p. 24-26) $C(I_s, E)$ is a Fréchet space.

THEOREM 1. *For an arbitrary $x \in E$, $B \in \mathscr{U}$, and $K \in \mathscr{V}$ there exists a unique function $y_{(x,B,K)} \in C(I_s, E)$ such that*

$$y_{(x,B,K)}(t) = K(t, s)x + \int_s^t K(t, \tau)B(\tau)y_{(x,B,K)}(\tau)d\tau$$

for all $t \in I_s$.

Proof. Let $x \in E$, $B \in \mathscr{U}$, and $K \in \mathscr{V}$. Define mappings T and G as follows: for $y \in C(I_s, E)$,

$$(Ty)(t) = \exp\big(-U(t)\big)y(t),$$

$$(Gy)(t) = \exp\big(-U(t)\big)\Big(K(t, s)x + \int_s^t K(t, \tau)B(\tau)y(\tau)d\tau\Big).$$

For $u, v \in C(I_s, E)$ and $s \leqslant t \leqslant n$, we have

$$\Big\|\int_s^t K(t,\tau)B(\tau)u(\tau)d\tau - \int_s^t K(t,\tau)B(\tau)v(\tau)d\tau\Big\|$$

$$\leqslant \int_s^t |B(\tau)|\exp\Big(\int_\tau^t \omega(\sigma)d\sigma\Big)\|u(\tau)-v(\tau)\|d\tau$$

$$= \int_s^t |B(\tau)|\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)\exp\Big(q\int_s^\tau |B(\sigma)|d\sigma\Big)\|(Tu)(\tau)-(Tv)(\tau)\|d\tau$$

$$\leqslant \exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)\sup_{s\leqslant t\leqslant n}\|(Tu)(t)-(Tv)(t)\|\int_s^t |B(\tau)|\exp\Big(q\int_s^\tau |B(\sigma)|d\sigma\Big)d\tau$$

$$< q^{-1}\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)\exp\Big(q\int_s^t |B(\sigma)|d\sigma\Big)p_n(Tu-Tv)$$

and it follows that $p_n(Gu-Gv) \leqslant q^{-1}p_n(Tu-Tv)$ for each $n \geqslant 1$. Therefore, Proposition 1 applies to the mappings T, G, and the space $C(I_s, E)$, which proves our theorem.

Here we use the notions of $\mathscr{L}^*$-space, the $\mathscr{L}^*$-product of $\mathscr{L}^*$-spaces, and a continuous mapping of the $\mathscr{L}^*$-space into the $\mathscr{L}^*$-space (see, e.g., [5]).

The set $\mathscr{U}$ will be considered as an $\mathscr{L}^*$-space endowed with the following convergence: (B_n) is a *convergent sequence* if

$$\sup_{n\geqslant 1}\sup_{t\in\Omega}|B_n(t)| < \infty$$

on compact subsets Ω of I_s and $\big(B_n(t)y(t)\big)$ converges uniformly on compact subsets of I_s for each $y \in C(I_s, E)$.

For example, $\mathscr{U}$ endowed with the *almost uniform convergence* (i.e., uniform convergence on every compact subset of I_s) is an $\mathscr{L}^*$-space satisfying the above conditions. Indeed, let Ω be a compact set of I_s and

$$\lim_{n\to\infty}\sup_{t\in\Omega}|B_n(t)-B_0(t)| = 0.$$

Then

$$\lim_{n\to\infty}\sup_{t\in\Omega}\|B_n(t)x-B_0(t)x\| = 0 \quad \text{for each } x \in E,$$

and therefore $\big(B_n(t)\big)$ is uniformly bounded for $t \in \Omega$ and $n \geqslant 1$. Further, by Lemma 3.4 in [4],

$$\lim_{n\to\infty}\sup_{t\in\Omega}\|B_n(t)y(t)-B_0(t)y(t)\| = 0 \quad \text{for every } y\in C(I_s, E),$$

so we are done.

A sequence (K_n) of elements of $\mathscr{V}$ is called *convergent* if $\big(K_n(t,\tau)y(\tau)\big)$ converges uniformly on $\Omega\times\Omega$ for each compact subset Ω of I_s and each $y\in C(I_s, E)$. The set $\mathscr{V}$ endowed with this convergence is an $\mathscr{L}^*$-space.

THEOREM 2. *Let $y_{(x,B,K)}$ be as in Theorem 1. Then the transformation $(x, B, K)\mapsto y_{(x,B,K)}$ from an $\mathscr{L}^*$-product $E\times\mathscr{U}\times\mathscr{V}$ into $C(I_s, E)$ is continuous.*

Proof. Let Ω be a compact subset of I_s. Without loss of generality we may assume that $s\in\Omega$. Moreover, denote by $C(\Omega, E)$ the Banach space of all continuous functions from Ω to E with the usual supremum norm $|||\cdot|||$.

Assume that

$$\lim_{n\to\infty}\|x_n-x_0\| = 0, \qquad \lim_{n\to\infty}B_n = B_0, \qquad \text{and} \qquad \lim_{n\to\infty}K_n = K_0,$$

where $(x_m, B_m, K_m)\in E\times\mathscr{U}\times\mathscr{V}$ for $m = 0, 1, \ldots$ Let

$$C' = \sup_{n\geqslant 1}\sup_{t\in\Omega}|B_n(t)|.$$

Define mappings T and G_m by

$$(Ty)(t) = \exp(-rt)y(t),$$

$$(G_my)(t) = \exp(-rt)\Big(K_m(t,s)x_m+\int_s^t K_m(t,\tau)B_m(\tau)y(\tau)\,d\tau\Big) \qquad (m = 0, 1, \ldots)$$

for $y\in C(\Omega, E)$, where

$$r > C'\sup_{t\in\Omega}\exp\Big(\int_s^t\omega(\sigma)\,d\sigma\Big).$$

For $y\in C(\Omega, E)$, $t\in\Omega$, and $n\geqslant 1$ we obtain

$$\|(G_ny)(t)-(G_0y)(t)\|$$

$$\leqslant \|K_n(t,s)x_n-K_0(t,s)x_0\| + \int_s^t\|K_n(t,\tau)B_n(\tau)y(\tau)-K_0(t,\tau)B_0(\tau)y(\tau)\|\,d\tau$$

$$\leqslant \|x_n-x_0\|\,|K_n(t,s)|+\|K_n(t,s)x_0-K_0(t,s)x_0\|+$$

$$+\int_s^t |K_n(t,\tau)|\,\|B_n(\tau)y(\tau)-B_0(\tau)y(\tau)\|\,d\tau+$$

$$+\int_s^t \|K_n(t,\tau)B_0(\tau)y(\tau)-K_0(t,\tau)B_0(\tau)y(\tau)\|\,d\tau$$

$$\leqslant \|x_n-x_0\|\sup_{t\in\Omega}\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)+\sup_{t,\tau\in\Omega}\|K_n(t,\tau)x_0-K_0(t,\tau)x_0\|+$$

$$+\sup_{t\in\Omega}\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)\sup_{t\in\Omega}\|B_n(t)y(t)-B_0(t)y(t)\|\sup_{t\in\Omega}|t-s|+$$

$$+\sup_{t,\tau\in\Omega}\|K_n(t,\tau)B_0(\tau)y(\tau)-K_0(t,\tau)B_0(\tau)y(\tau)\|\sup_{t\in\Omega}|t-s|,$$

and therefore $|||G_ny-G_0y|||\to 0$ as $n\to\infty$. Obviously,

$$G_n[C(\Omega,E)]\subset T[C(\Omega,E)]=T[C(\Omega,E)]$$

and

$$|||G_nu-G_nv|||\leqslant r^{-1}C'\sup_{t\in\Omega}\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big)\,|||Tu-Tv|||$$

for $n\geqslant 1$ and $u,v\in C(\Omega,E)$.

Consequently, by Proposition 1, there exists a unique $y_m\in C(\Omega,E)$ $(m=0,1,\ldots)$ such that $y_{(x_m,B_m,K_m)}|_\Omega=y_m$ and $|||y_n-y_0|||\to 0$ as $n\to\infty$. This completes our proof.

Now, we study the existence of solutions of the nonlinear equation $(++)$ with f satisfying the assumption (c). The results will follow from the fixed point theorem of Schauder type given in Section 2 as Proposition 2.

THEOREM 3. *Let $x\in E$, $B\in\mathscr{U}$, and $K\in\mathscr{V}$. Assume that for each $h>0$ there exists a compact subset Z_h of E such that $f\big[[0,h]\times E\big]\subset Z_h$. Then equation $(++)$ has at least one solution in $C(I_s,E)$.*

Proof. Define mappings T, Q, and F by

$$(Ty)(t)=\exp\big(-U(t)\big)y(t),$$

$$(Qy)(t)=\int_s^t K(t,\tau)f\big(\tau,\exp\big(U(\tau)\big)y(\tau)\big)d\tau,$$

$$F(u,v)(t)=\exp\big(-U(t)\big)\Big(K(t,s)x+\int_s^t K(t,\tau)B(\tau)u(\tau)d\tau+$$

$$+\int_s^t K(t,\tau)f\big(\tau,\exp\big(U(\tau)\big)v(\tau)\big)d\tau\Big)$$

for all $y,u,v\in C(I_s,E)$.

Arguments similar to those in the proof of Theorem 1 imply that

$$p_n\big(F(u,y)-F(v,y)\big) \leqslant q^{-1}p_n(Tu-Tv) \tag{1}$$

and

$$p_n\big(F(y,u)-F(y,v)\big) \leqslant C_n p_n(Qu-Qv) \tag{2}$$

for all $n \geqslant 1$ and $u, v, y \in C(I_s, E)$, where $C_n = \exp\big(-U(n)\big)$. Since $y(\cdot) \mapsto f\big(\cdot, y(\cdot)\big)$ is a continuous mapping of $C(I_s, E)$ into itself, Q is also continuous.

Now, we prove that the set $\overline{Q[C(I_s, E)]}$ is compact.

Let us put

$$V_t = \Big\{\int_s^t K(t,\tau) f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big)d\tau : y \in C(I_s, E)\Big\} \quad \text{for } t \geqslant s.$$

Further, let Z_t be a compact set such that $f\big[[0,t]\times E\big] \subset Z_t$. Since $(\tau, x) \mapsto K(t,\tau)x$ is continuous on $[s,t]\times E$, $\{K(t,\tau)x : s \leqslant \tau \leqslant t, x \in Z_t\}$ is compact in E and, consequently, the set

$$W_t = \big\{K(t,\tau) f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big) : s \leqslant \tau \leqslant t, \ y \in C(I_s, E)\big\}$$

is conditionally compact. For vector-valued functions the integral mean-value theorem may be stated as

$$\int_a^b y(\tau)\,d\tau \in (b-a)\overline{\operatorname{conv}}\big(\{y(\tau) : a \leqslant \tau \leqslant b\}\big),$$

where $\overline{\operatorname{conv}}(A)$ denotes the closed convex hull of A. Therefore,

$$V_t \subset (t-s)\overline{\operatorname{conv}}(W_t),$$

and by the Mazur theorem (see [3], p. 163-164, or [7]) the set $\overline{V}_t$ is compact.

Let Ω be a compact subset of I_s. Suppose that $\Omega = [s, t]$. We put

$$I_1 = \int_{t_1}^{t_2} |K(t_2,\tau)| \big\| f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big)\big\| d\tau,$$

$$I_2 = \int_s^{t_1} \Big\| \big(K(t_2,\tau) - K(t_1,\tau)\big) f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big)\Big\| d\tau$$

for $y \in C(I_s, E)$ and $t_1, t_2 \in \Omega$. We have

$$\big\{f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big) : s \leqslant \tau \leqslant t, \ y \in C(I_s, E)\big\} \subset Z$$

for some compact subset Z of E. Hence $I_1 \leqslant C|t_2 - t_1|$ for $y \in C(I_s, E)$, where

$$C = \operatorname{diam}(Z)\exp\Big(\int_{\Omega} \omega(\sigma)\,d\sigma\Big).$$

Since the function $(t, \tau, x) \mapsto K(t, \tau)x$ is uniformly continuous on $\Omega \times \Omega \times Z$, for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$I_2 \leqslant \varepsilon |t_1 - s| \leqslant \operatorname{diam}(\Omega)\varepsilon$$

if $|t_1 - t_2| < \delta$ and $y \in C(I_s, E)$. From the inequality

$$\Big\|\int_s^{t_2} K(t_2, \tau) f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big)d\tau - \int_s^{t_1} K(t_1, \tau) f\big(\tau, \exp\big(U(\tau)\big)y(\tau)\big)d\tau\Big\| \leqslant I_1 + I_2$$

for $y \in C(I_s, E)$ and $t_1, t_2 \in \Omega$ it follows that the set $Q[C(I_s, E)]$ consists of equicontinuous functions on Ω.

Finally, by Ascoli-Arzela theorem ([3], p. 34) the set $Q[C(I_s, E)]$ is conditionally compact in $C(I_s, E)$, and all the assumptions of Proposition 2 are satisfied. Hence there exists $y \in C(I_s, E)$ such that $F(y, Ty)(t) = (Ty)(t)$ for each $t \geqslant s$. Thus the proof is complete.

THEOREM 4. *Assume that $B \in \mathscr{U}$, $K \in \mathscr{V}$, and that the function f has the following properties:*

(i) *$f[V]$ is totally bounded in E whenever V is bounded in $I_0 \times E$;*

(ii) *there is a continuous function $h\colon I_s \times I_0 \to I_0$ monotonically non-decreasing in the second variable and such that $\|f(t, x)\| \leqslant h(t, \|x\|)$ for $t \geqslant s$ and $x \in E$;*

(iii) *for every $r \geqslant 0$ there exists a locally bounded function $\varphi_r\colon I_s \to I_0$ such that*

$$\varphi_r(t) \geqslant r\exp\Big(\int_s^t \omega(\sigma)\,d\sigma\Big) + \int_s^t \exp\Big(\int_\tau^t \omega(\sigma)\,d\sigma\Big)\Big(|B(\tau)|\varphi_r(\tau) + h\big(\tau, \varphi_r(\tau)\big)\Big)d\tau$$

in the set I_s.

Then for every $x \in E$ there exists a solution $y \in C(I_s, E)$ of equation $(++)$ and $\|y(t)\| \leqslant \varphi_{r_0}(t)$ for $t \geqslant s$, where $r_0 = \|x\|$.

Proof. Let us put

$$X = \{y \in C(I_s, E)\colon \|y(t)\| \leqslant \varphi_{r_0}(t) \text{ for } t \geqslant s\},$$

$$Y = \big\{y \in C(I_s, E)\colon \|y(t)\| \leqslant \varphi_{r_0}(t)\exp\big(-U(t)\big) \text{ for } t \geqslant s\big\}$$

and define the mappings T, Q, and F as in the proof of Theorem 3. Obviously, Y is a closed convex set, $T[X]$ is complete, Q is continuous, and (1) and (2) hold. Applying condition (i) and modifying the reasoning from the proof of Theorem 3 we infer that the set $Q[Y]$ is compact.

Let us put

$$y(t) = K(t, s)x + \int_s^t K(t, \tau)B(\tau)y_1(\tau)d\tau + \\ + \int_s^t K(t, \tau)f\big(\tau, \exp\big(U(\tau)\big)y_2(\tau)\big)d\tau \quad \text{for } t \geqslant s,$$

where $y_1 \in X$ and $y_2 \in Y$. Then $Ty = F(y_1, y_2)$ and, for any $t \in I_s$, we have

$$\|y(t)\| \leqslant \|x\|\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big) + \int_s^t \exp\Big(\int_\tau^t \omega(\sigma)d\sigma\Big)|B(\tau)|\,\|y_1(\tau)\|d\tau + \\ + \int_s^t \exp\Big(\int_\tau^t \omega(\sigma)d\sigma\Big)\Big\|f\big(\tau, \exp\big(U(\tau)\big)y_2(\tau)\big)\Big\|d\tau \\ \leqslant \|x\|\exp\Big(\int_s^t \omega(\sigma)d\sigma\Big) + \int_s^t \exp\Big(\int_\tau^t \omega(\sigma)d\sigma\Big)|B(\tau)|\varphi_{r_0}(\tau)d\tau + \\ + \int_s^t \exp\Big(\int_\tau^t \omega(\sigma)d\sigma\Big)\, h\big(\tau, \varphi_{r_0}(\tau)\big)d\tau \leqslant \varphi_{r_0}(t),$$

and therefore $y \in X$. Consequently, $F[X \times Y] \subset T[X]$ and, by Proposition 2, we obtain the assertion.

5. Final remarks. We give some remarks about applications of our results to the theory of infinite systems of differential equations (cf. [6]).

Notice, for example, that Theorems 1 and 2 can be simply applied to the case of the Banach space l^1 of all scalar sequences (x_n) such that

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

Consider the infinite system

$$y_i'(t) = \sum_{j=1}^{\infty} a_{ij}(t)y_j(t), \qquad i = 1, 2, \ldots,$$

with initial conditions $y_i(0) = x_i$ for $i \geqslant 1$. Denote by $\mathfrak{M}$ the set of all matrices $A = [a_{ij}]$ $(i, j = 1, 2, \ldots)$ such that each a_{ij} is a continuous function on I_0, $a_{ii}(t) \equiv 0$ on I_0 for each $i \geqslant 1$, $\sum_{i=1}^{\infty} |a_{ij}(t)|$ converges uniformly on compact subsets of I_0 for each $j \geqslant 1$, and the function

$$t \mapsto \sup_{j\geqslant 1} \sum_{i=1}^{\infty} |a_{ij}(t)|$$

is locally bounded on I_0. This set can be considered with the almost uniform convergence, i.e.,

$$\lim_{n\to\infty} [a_{ij}^{(n)}] = [a_{ij}^{(0)}] \quad \text{in } \mathfrak{M}$$

if and only if

$$\lim_{n\to\infty} \sup_{t\in\Omega} \sup_{j\geqslant 1} \sum_{i=1}^{\infty} |a_{ij}^{(n)}(t) - a_{ij}^{(0)}(t)| = 0$$

on compact subsets Ω of I_s.

Now, using Theorems 1 and 2 we obtain the following corollary: *For each $x = (x_n)$ in l^1 and $A = [a_{ij}]$ in $\mathfrak{M}$, the above infinite system has a unique solution $y_{(x,A)} = (y_1, y_2, \ldots)$ in the space $C(I_s, l^1)$. Moreover, $(x, A) \mapsto y_{(x,A)}$ is a continuous mapping from $l^1 \times \mathfrak{M}$ into $C(I_s, l^1)$.*

Finally, notice that the second part of this corollary may be used to show that the solution of our infinite system is a limit of solutions to finite systems obtained from it by truncation. The results of such a type have been obtained by the application of Gronwall's inequality (see [6], Theorem 3.1).

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