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A CLASS OF COMPLEX HYPERSURFACES

BY

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The study of complex hypersurfaces was initiated by Smyth [9]. His first paper classified the Kähler-Einstein manifolds which occur as hypersurfaces in complex space forms. The same classification was obtained by Takahashi [11] and Nomizu and Smyth [5] under the weaker assumption of parallel Ricci tensor.

In this paper*, we discuss the still weaker condition that *each curvature transformation commutes with the Ricci tensor*. For notational reasons which will become clear, we write this condition in the form $RS = 0$. Our main theorem will be the following

THEOREM 1. *The complete Kähler manifolds with $RS = 0$ which occur as hypersurfaces in complex space forms of holomorphic sectional curvature $c \neq 0$ are*

(i) *the complex projective space $P^n(c)$ and the complex quadric Q^n in $P^{n+1}(c)$;*

(ii) *the disk $D^n(c)$ of holomorphic sectional curvature $c < 0$ in $D^{n+1}(c)$.*

Some questions still remain unanswered when $c = 0$. However, we show that C^n is the only complex hypersurface in C^{n+1} which has $RS = 0$ and constant scalar curvature.

The real version of this theorem has been treated by Tanno [12] and the author [8]. The latter makes use of results of the author [7] concerning the related condition $RR = 0$.

All the concepts mentioned in this section are defined in the sequel or can be found in Kobayashi and Nomizu [3].

1. Preliminaries. Let M be a Riemannian manifold of complex dimension n with Riemannian metric g. The *curvature tensor* R of M is a tensor field of type (1,3). For each pair of tangent vectors (X, Y) at a point of M, $R(X, Y)$ is a skew-symmetric endomorphism of the

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tangent space. We call $R(X, Y)$ a *curvature transformation.* If X and Y are vector fields, $R(X, Y)$ may be extended uniquely to a derivation of the algebra of tensor fields so as to commute with all contractions. The resulting derivation, also denoted by $R(X, Y)$, may be expressed in terms of the Levi-Cevità connection ∇ by

$$R(X, Y)\cdot T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T, \tag{1}$$

where T is a tensor field of any type (r, s). We denote by RT the tensor field of type $(r, s+2)$ defined by

$$RT(X_1, X_2, \ldots, X_s, X, Y) = \big(R(X, Y)\cdot T\big)(X_1, X_2, \ldots, X_s).$$

We will be concerned in this paper with only two cases, namely when T is of type (1,3) (the curvature tensor) and when T is of type (1,1) (the Ricci tensor).

In the (1,3) case, we verify that

$$\big(R(X, Y)\cdot T)\big)(U, V) = [R(X, Y), T(U, V)] - T\big(R(X, Y)U, V\big) - \\ - T\big(U, R(X, Y)V\big).$$

and, in the (1,1) case,

$$\big(R(X, Y)\cdot T\big)Z = R(X, Y)(TZ) - T\big(R(X, Y)Z\big),$$

i.e. $R(X, Y)\cdot T = [R(X, Y), T]$. Here [,] denotes the commutator of two linear transformations.

Thus, the condition $RT = 0$, can be written as

$$[R(X, Y), T(U, V)] = T\big(R(X, Y)U, V\big) + T\big(U, R(X, Y)V\big) \tag{2}$$

or

$$[R(X, Y), T] = 0 \tag{3}$$

for the cases in which we are interested.

The Ricci tensor of M is the tensor of type (1,1) defined by

$$SX = \sum_i R(X, e_i)e_i, \tag{4}$$

where e_i is an orthonormal basis. It is also characterized by the fact that $g(SX, Y)$ is the trace of the linear mapping $Z \to R(Z, X)Y$.

S is a symmetric endomorphism of each tangent space. The smooth function $s = \text{trace } S$ is called the *scalar curvature.*

A Riemannian manifold is said to be *Einstein* if S is a scalar multiple of the identity. If the (real) dimension of M is greater than 2, this scalar multiple must be constant.

A tensor field T is said to be *parallel* if $\nabla_X T = 0$ for all vector fields X. This is written $\nabla T = 0$. Thus the Ricci tensor of an Einstein space is parallel. If $\nabla S = 0$, we see that s must be constant since

$$\nabla_X(\mathrm{trace}\, S) = \mathrm{trace}(\nabla_X S).$$

Furthermore, in view of (1), $RT = 0$ whenever $\nabla T = 0$.

2. Complex hypersurfaces. Let M be a connected complex manifold of dimension n holomorphically immersed in Kähler manifold $\tilde{M}$ of dimension $n+1$. We call M a *complex hypersurface* in $\tilde{M}$. The complex structure J and the Kähler metric g of $\tilde{M}$ induce, respectively, a complex structure and a Kähler metric on M. We denote these induced objects by the same letters.

We recall some basic ideas from [5] and [9]. For each $x_0 \in M$, we choose a smooth field of unit normals ξ defined in a neighborhood U of x_0. Denoting by $\tilde{\nabla}$ the Kählerian connection on $\tilde{M}$, we have, for vector fields tangent to M in U,

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi + g(JAX, Y)J\xi$$

and

$$\tilde{\nabla}_X \xi = -AX + s(X)J\xi,$$

where A is a symmetric tensor field of type (1,1) on U, called the *second fundamental form*, and ∇ is the induced Kähler connection on M. It is easy to show that

$$AJ = -JA. \tag{5}$$

The curvature tensor R of M is expressed by the Gauss equation

$$R(X, Y) = \tilde{R}(X, Y) + AX \wedge AY + JAX \wedge JAY, \tag{6}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{M}$. Although A depends on our choice of ξ, it is not difficult to show that A^2 is independent of this choice and so is defined on all of M as a tensor field of type (1,1).

A two-dimensional subspace π of the (real) tangent space is called a *holomorphic plane* if there is a unit vector X such that X and JX span π. A Kähler manifold $\tilde{M}$ is said to have *constant holomorphic sectional curvature* c if the number

$$K(\pi) = g\big(\tilde{R}(X, JX)JX, X\big)$$

is equal to c for every holomorphic plane π at every point of M. It is well known that this is true if and only if

$$\tilde{R}(X, Y) = \frac{c}{4}\big(X \wedge Y + JX \wedge JY + 2g(X, JY)J\big) \tag{7}$$

holds for all tangent vectors X and Y. We are using the symbol $\wedge$ in the following sense: $u\wedge v$ is the skew-symmetric linear transformation defined by

$$(u\wedge v)w = g(v, w)u - g(u, w)v.$$

3. The standard examples. A complex space form is a complete, simply-connected Kähler manifold of constant holomorphic sectional curvature. For each real number c, there is (up to holomorphic isometry) exactly one complex space form in every dimension with curvature c.

The complex space forms of curvature c are denoted by $P^n(c)$, C^n and $D^n(c)$ depending on whether c is positive, zero or negative. $P^n(1)$ is the complex projective space endowed with the standard Fubini-Study metric. C^n is the complex Euclidean space. $D^n(-1)$ is the unit ball in C^n with the Bergman metric. $P^n(c)$ (respectively, $D^n(c)$) is obtained by multiplying the metric of $P^n(1)$ (respectively, $D^n(-1)$) by a positive constant. Details may be found in [3], p. 159-163.

$P^n(c)$, C^n and $D^n(c)$ occur naturally as totally geodesic complex hypersurfaces in $P^{n+1}(c)$, C^{n+1} and $D^{n+1}(c)$, respectively. In each case, just set one coordinate equals 0. In the case of projective space, there is another interesting complex hypersurface defined to be

$$\{Z\,|\,Z_0^2+Z_1^2+\ldots+Z_{n+1}^2 = 0\} \qquad (Z_i \text{ homogeneous coordinates}).$$

This space is called the *complex quadric* Q^n.

In each of the standard examples, the second fundamental form A is very simple. It is identically zero in the totally geodesic cases and for the quadric $A^2 = cI/4$, where I is the identity.

4. The curvature operators. Let M be a complex hypersurface in a Kähler manifold $\tilde{M}$. For any $x\in M$, the tangent space to M at x is a $2n$-dimensional real-vector space with inner product g and complex structure J. The facts that $g(JX, JY) = g(X, Y)$ and $AJ = -JA$ lead to the existence of an orthonormal basis of eigenvectors of A which takes the form $\{e_i, Je_i\}_{i=1}^n$. For details, see Lemma 1 of [9]. We may choose the ordering so that, for $1\leqslant i\leqslant n$,

$$Ae_i = \lambda_i e_i \quad \text{and} \quad AJe_i = -\lambda_i Je_i, \quad \lambda_i\geqslant 0. \tag{8}$$

If $\tilde{M}$ has constant holomorphic sectional curvature c, (6) and (7) imply that

$$R(e_i, e_j) = \left(\frac{c}{4}+\lambda_i\lambda_j\right)(e_i\wedge e_j+Je_i\wedge Je_j), \tag{9}$$

$$R(e_i, Je_j) = \left(\frac{c}{4}-\lambda_i\lambda_j\right)(e_i\wedge Je_j-Je_i\wedge e_j)-\frac{c}{4}\delta_{ij}J. \tag{10}$$

5. The condition $RS = 0$. It is an easy consequence of the Gauss equation (6) that

$$S = \frac{n+1}{2} cI - 2A^2.$$

Thus, using the basis of (8), we have

$$Se_i = \left(\frac{n+1}{2} c - 2\lambda_i^2\right) e_i, \qquad S(Je_i) = \left(\frac{n+1}{2} c - 2\lambda_i^2\right) Je_i.$$

The following theorem characterizes the complex hypersurfaces with $RS = 0$ in terms of A:

THEOREM 2. *Let M be a complex hypersurface in a space of constant holomorphic sectional curvature c. Then $RS = 0$ on M if and only if one of the following is true*:

(i) $c \neq 0$, *A^2 is a multiple of I*;

(ii) $c = 0$ *and the non-zero eigenvalues of A^2 are equal.*

Remark. This theorem shows that when $c \neq 0$ $RS = 0$ if and only if M is Einstein.

Proof. Suppose $RS = 0$. Then, for $i \neq j$,

$$R(e_i, e_j)(Se_j) = \left(\frac{c}{4} + \lambda_i\lambda_j\right)\left(\frac{n+1}{2} c - 2\lambda_j^2\right) e_i,$$

$$S\big(R(e_i, e_j)e_j\big) = \left(\frac{c}{4} + \lambda_i\lambda_j\right)\left(\frac{n+1}{2} c - 2\lambda_i^2\right) e_i.$$

Hence

$$\left(\frac{c}{4} + \lambda_i\lambda_j\right)(\lambda_i^2 - \lambda_j^2) = 0. \tag{11}$$

Similarly,

$$R(e_i, Je_j)(Se_j) = -\left(\frac{c}{4} - \lambda_i\lambda_j\right)\left(\frac{n+1}{2} c - 2\lambda_j^2\right) Je_i,$$

$$S\big(R(e_i, Je_j)e_j\big) = -\left(\frac{c}{4} - \lambda_i\lambda_j\right)\left(\frac{n+1}{2} c - 2\lambda_i^2\right) Je_i.$$

Hence

$$\left(\frac{c}{4} - \lambda_i\lambda_j\right)(\lambda_i^2 - \lambda_j^2) = 0. \tag{12}$$

Recalling that each λ_i is non-negative, we immediately see that if $c \neq 0$, then all of the λ_i are equal. Thus the identity

$$A^2 = \frac{\operatorname{trace} A^2}{2n} I$$

holds. On the other hand, when $c = 0$, (11) and (12) are the same and they clearly imply that all the non-zero λ_i are equal.

Conversely, if S is a multiple of the identity, then S commutes with every curvature operator and $RS = 0$. Now suppose that $c = 0$ and all the non-zero eigenvalues of A^2 are equal. Working at a particular point $x \in M$, let T_0 and T_1 denote the eigenspaces of A^2 and hence of S. We must check that $R(X, Y)\cdot S = 0$ for all X and Y. If either X or Y is in T_0, then $R(X, Y) = 0$. Thus we may suppose X and Y are in T_1. By the Bianchi identity

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

we see that $R(X, Y)T_0 \subseteq T_0$. Since $R(X, Y)$ is skew-symmetric, it must also map T_1 into T_1. But S is a multiple of the identity as a mapping of T_1 into T_1. Thus $R(X, Y)$ commutes with S. This completes the proof of the theorem.

We shall see later that when $c = 0$, the Einstein condition completely determines the hypersurface M and its curvature tensor will satisfy $\nabla R = 0$ and hence $RR = 0$. Thus the conditions $RR = 0$ and $RS = 0$ are equivalent when $c \neq 0$. However, when $c = 0$, it is not known whether these conditions are equivalent. We have the following

PROPOSITION 3. *Suppose $c = 0$ and $RR = 0$ in the preceeding theorem. Then the rank of A^2 is at most 2 at any point.*

Proof. We begin by substituting in (2) to obtain the following identity for $i \neq j$:

$$[R(e_i, e_j), R(e_i, Je_j)] = R\big(R(e_i, e_j)e_i, Je_j\big) + R\big(e_i, R(e_i, e_j)Je_j\big).$$

The left-hand side is equal to

$$-\lambda_i^2\lambda_j^2[e_i\wedge e_j + Je_i\wedge Je_j, e_i\wedge Je_j - Je_i\wedge e_j] = -\lambda_i^2\lambda_j^2(e_i\wedge Je_i - e_j\wedge Je_j).$$

The first summand is

$$\lambda_i\lambda_j R(-e_j, Je_j) = \lambda_i\lambda_j(2\lambda_j^2)(e_j\wedge Je_j).$$

The second summand is

$$\lambda_i\lambda_j(-2\lambda_i^2)(e_i\wedge Je_i).$$

Comparing coefficients of the linearly independent elements $e_i\wedge Je_i$ and $e_j\wedge Je_j$, we get the two identities $\lambda_i^2\lambda_j^2 = 2\lambda_j^3\lambda_i = 2\lambda_i^3\lambda_j$. This gives $\lambda_i\lambda_j^2(2\lambda_j - \lambda_i) = 0$. Hence, if $\lambda_i\lambda_j \neq 0$, we must have $\lambda_i = 2\lambda_j$. But since $RS = 0$, we have also $\lambda_i = \lambda_j$. This is a contradiction. We conclude that λ_i is non-zero for at most one i.

6. The classification.

THEOREM 4. *For a complex hypersurface of complex dimension* $n > 1$ *in a space of constant holomorphic sectional curvature* $c \neq 0$, *the following are equivalent:*

(1) $RR = 0$ *on* M;

(2) $RS = 0$ *on* M;

(3) M *is Einstein*;

(4) S *is parallel, i.e.* $\nabla S = 0$;

(5) R *is parallel, i.e.* $\nabla R = 0$;

(6) M *is totally geodesic or* $c > 0$ *and* M *is locally the quadric* Q^n (*globally, if* M *is complete and* $\tilde{M} = P^{n+1}(c)$).

Proof. The equivalence of (2) and (3) is given by Theorem 2. Theorem 4 of Nomizu and Smyth [5] says that (4) and (6) are equivalent. The standard examples are all Riemannian symmetric spaces. Thus (6) implies (5). Thus, we have the following string of implications:

$$(2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (6) \Rightarrow (5) \Rightarrow (1) \Rightarrow (2).$$

The last implication is a result of the fact that S is a contraction of R and every $R(X, Y)$ commutes with contractions. This completes the proof.

7. The case of constant scalar curvature. As mentioned in Theorem 4, Nomizu and Smyth classified the complex hypersurfaces with parallel Ricci tensor. They proved that if $\nabla S = 0$ on M, then

(i) when $c = 0$, M is totally geodesic;

(ii) when $c \neq 0$, M is totally geodesic or is locally the complex quadric, the latter arising only when $c > 0$.

Kobayashi [2] weakened the assumption in case $c > 0$ to compactness and constant scalar curvature. We now observe that Theorem 2 allows the weakening of (i) as follows:

PROPOSITION 5. *A complex hypersurface in* C^{n+1} *having* $RS = 0$ *and constant scalar curvature must be totally geodesic.*

Proof. Let s denote the scalar curvature. Choose a point x, where the rank of S is maximal. Then the rank of S is constant in some neighborhood of this point. If the rank is k, then the non-zero eigenvalue $-2\lambda^2$ satisfies the relation $-2\lambda^2 k = s$.

Since s and k are constant near x, so is λ^2. We now appeal to a theorem of Smyth [10], which says that if the eigenvalues of A^2 are constant in value and multiplicity, then they are all zero. This shows that $s = 0$ in a neighborhood of x and hence $\lambda^2 = 0$ on all of M. Thus M is totally geodesic, i.e. a piece of a complex hyperplane.

8. Concluding remarks. The classification of complex hypersurfaces with $RR = 0$ was done by a longer method in an unpublished portion of the author's thesis [6]. These results were announced in the notices of the American Mathematical Society, August, 1968. The part of this pertaining to the case $c = 0$ (Proposition 3) takes on added significance in light of a subsequent theorem of Abe [1] which follows.

THEOREM (Abe). *Let M be a complete complex hypersurface in C^{n+1} with rank $A \leqslant 2$ at each point. Then M is cylindrical, i.e. M is the product of an $(n-1)$-dimensional complex hyperplane C^{n-1} and a complex curve in a 2-dimensional complex plane orthogonal to C^{n-1} in C^{n+1}.*

We remark that $RR = 0$ whenever rank $A \leqslant 2$. Hence, we have the following consequence of Proposition 3 and Abe's theorem:

THEOREM 6. *Let M be a complete complex hypersurface in C^{n+1}. Then $RR = 0$ on M if and only if M is cylindrical.*

The following question remains unanswered: Can $RR = 0$ be replaced by $RS = 0$ in Theorem 6? (**P 808**)

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