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CARDINALS AND ITERATIONS OF HOD

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1. Introduction. The sequence of inner models HOD_n is nonincreasing, and therefore the sequence of classes $(\mathrm{Card})^{\mathrm{HOD}_n}$ is nondecreasing. Let $\varkappa$ be an aleph. Then $\varkappa$ is a cardinal in every HOD_n and there exist ordinals α_n such that $\varkappa = (\omega_{\alpha_n})^{\mathrm{HOD}_n}$. The sequence $\langle \alpha_n \colon n < \omega \rangle$ is nondecreasing. We ask whether such a sequence of indices must be nearly constant or whether it can strictly increase infinitely many times. The answer is given by the following theorem:

THEOREM. *Let M be a countable standard model (c.s.m.) for* ZF + V = L, *let $f \in M$,*

$$f\colon \omega \to \mathrm{On}, \quad \mathrm{rng}(f) \subseteq \{1\} \cup \{\alpha + 2 \colon \alpha \in \mathrm{On}\},$$
$$f(n) \leqslant f(m) \quad \text{for } n \leqslant m.$$

Then there exists a c.s.m. N for ZFC, *which is a generic extension of M, such that*

$$N \models (n)_\omega \big((\omega_{f(n)})^{\mathrm{HOD}_n} = \omega_{f(0)} \big).$$

To prove the Theorem we use a modification of a method described in the paper by McAloon([1]).

2. Construction. Let M and f satisfy the assumptions of the Theorem. Let

$$\lambda = \sup\{f(n) \colon n < \omega\}.$$

We want to add a sequence of functions $\langle g_n \colon n < \omega \rangle$ such that g_n collapses $\omega^{\mathrm{L}}_{\lambda+1}$ on $\omega^{\mathrm{L}}_{f(n)-1}$ for $n < \omega$, and $g_k \in \mathrm{HOD}_n$ iff $k \leqslant n$ for $n, k < \omega$.

Let $\varkappa$, μ be cardinals. By $C(\varkappa, \mu)$ we denote the notion of forcing "collapsing" μ on $\varkappa$. Its elements are functions of power $< \varkappa$, with the domain included in $\varkappa$ and the range included in μ.

DEFINITION 1.
$$C_n = C(\omega^{\mathrm{L}}_{f(n)-1}, \omega^{\mathrm{L}}_{\lambda+1}) \quad \text{for } n < \omega.$$

$$P = \prod_{n \in \omega} C_n, \quad P_k = \{p \in P \colon p(n) = \emptyset \text{ for } n < k\}.$$

([1]) K. McAloon, *On the sequence of* HOD_n *models*, Fund. Math. 82 (1974), pp. 85–93.

First note some useful properties of the notions of forcing defined above.

FACT 2. C_n *is* $\omega^{\mathrm{L}}_{f(n)-1}$ *closed*, $|C_n| = \omega^{\mathrm{L}}_{\lambda+1}$.

P *is* $\omega^{\mathrm{L}}_{f(0)-1}$ *closed*, $P = \omega^{\mathrm{L}}_{\lambda+1}$.

P_n *is* $\omega^{\mathrm{L}}_{f(n)-1}$ *closed*, $P_n = \omega^{\mathrm{L}}_{\lambda+1}$.

Let $\varkappa_0$ be the first cardinal larger than λ and such that $\omega_{\varkappa_0} = \varkappa_0$.

DEFINITION 3.

$$p\in Q \leftrightarrow \mathrm{func}(p)\ \&\ (p \subseteq \varkappa_0 \times 2)\ \&\ (\alpha)_{\varkappa_0}(\alpha \geqslant \lambda \rightarrow |p \restriction \omega^{\mathrm{L}}_{\alpha+2}|\omega^{\mathrm{L}}_{\alpha+2}),$$

and $\leqslant$ is the reversed inclusion.

Let $G\times H$ be $(P\times Q)$-generic over M. Then $M[G\times H] \models \mathrm{ZFC}$. We define, in $M[G\times H]$, some auxiliary sets and functions.

DEFINITION 4.

$$G_n = \{q\in C_n\colon\ p(n) \leqslant q \quad \text{for some } p\in G\}, \qquad g_n = \bigcup G_n,$$
$$A_\alpha = \big(\bigcup H\big)^{-1}(0) \cap (\omega^{\mathrm{L}}_{\alpha+2} - \omega^{\mathrm{L}}_{\alpha+1}).$$

Let us recall that x is a *good subset of a cardinal* $\varkappa$ if x is a cofinal subset of $\varkappa$ and x has no constructible cofinal subset. One can observe that g_n collapses $\omega^{\mathrm{L}}_{\lambda+1}$ on $\omega^{\mathrm{L}}_{f(n)-1}$ for $n<\omega$ and A_α is a good subset of $\omega^{\mathrm{L}}_{\alpha+2}$ for $\lambda \leqslant \alpha < \varkappa_0$. Let J be the Gödel pair function for ordinals.

DEFINITION 5.

$$C^0_m = \{\lambda + J(\alpha, \beta) * \omega + m + 2\colon\ g_m(\alpha) = \beta\},$$

$$C^{k+1}_m = \bigcup \{A_\alpha\colon\ \alpha\in C^k_m\}, \qquad B_n = \bigcup_{m=n}^{\infty} \bigcup_{k=0}^{m-n} C^k_m.$$

FACT 6. $\alpha\in B_{n+1} \leftrightarrow A_\alpha \subseteq B_n$.

DEFINITION 7. $N = \mathrm{L}[B_0]$.

3. Properties of N. Let us note that

$$\mathrm{L}[B_n] \models \omega^{\mathrm{L}}_{\lambda+2} = \omega^{\mathrm{L}}_{f(n)}.$$

This is true because

$$g_n = \{\langle\alpha, \beta\rangle \in \omega^{\mathrm{L}}_{f(n)-1} \times \omega^{\mathrm{L}}_{\lambda+1}\colon\ \lambda + J(\alpha, \beta) * \omega + m + 2 \in B_n\} \in \mathrm{L}[B_n].$$

All we need to complete the proof of the Theorem is the equality $\mathrm{L}[B_n] = \mathrm{HOD}_n$. We prove it by induction. For $n = 0$, see Definition 7. Here is the induction step.

FACT 8. $\mathrm{L}[B_{n+1}] = (\mathrm{HOD})^{\mathrm{L}[B_n]}$.

Since $\mathrm{L}[B_n]$ may be obtained as a generic extension of $\mathrm{L}[B_{n+1}]$ via a homogeneous notion of forcing, the inclusion

$$(\mathrm{HOD})^{\mathrm{L}[B_n]} \subseteq \mathrm{L}[B_{n+1}]$$

holds.

On the other hand, from Fact 6 we get

$$\alpha \in B_{n+1} \leftrightarrow L[B_n] \models \text{“}\omega_{\alpha+2} \text{ has a good subset”}.$$

It follows that $B_{n+1} \subseteq (\mathrm{HOD})^{L[B_n]}$, which implies the reverse inclusion. This completes the induction step and the proof of the Theorem is finished.

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