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A MULTIPLIER THEOREM FOR SCHRÖDINGER OPERATORS

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Let V be a nonnegative function on $\mathbf{R}^d$ such that the form

$$B(f) = \sum \|\partial_i f\|^2 + \|V^{1/2} f\|^2$$

is densely defined. This form is closed, so there exists a uniquely determined positive definite operator A such that $(Af, f) = B(f)$ and $D(A^{1/2}) = D(B)$. Let E be the spectral measure of A. If F is a bounded Borel measurable function we write

$$F(A)f = \int F(\lambda)\, dE(\lambda) f.$$

Let

$$F_t(x) = F(tx).$$

By the spectral theorem $F(A)$ is bounded on L^2. It is an interesting problem to give sufficient conditions on F and A which imply boundedness of $F(A)$ on L^p, $p \neq 2$. In the case $A = -\Delta$ the classical Hörmander multiplier theorem asserts that if for the Sobolev norm $\|\cdot\|_{H(q)}$ we have

$$\sup_{t>0} \|\varphi F_t\|_{H(q)} < \infty,$$

for some $q > d/2$, $\varphi \in C_c^\infty(\mathbf{R}_+)$, $\varphi \neq 0$, then $F(A)$ is bounded on L^p, $1 < p < \infty$. A similar result was obtained for A being a homogeneous sublaplacean on a homogeneous Lie group G by A. Hulanicki and E. M. Stein (cf. [2]) (with large q) and M. Christ [1] (q being half the homogeneous dimension of G).

It has been noticed that by transferring this and similar theorems to groups by means of representations one obtains multiplier theorems for Schrödinger operators with potentials which are sums of squares of polynomials.

On the other hand, methods of [3] yield multiplier theorems for some Schrödinger operators (e.g. if $d = 1$). To apply this or other standard methods one needs a Hölder condition on the kernels of the semigroup e^{tA} like

$$(*) \qquad |e^{tA}1(x) - e^{tA}1(x')| \leq Ct^{-\alpha/2}|x - x'|^\alpha$$

for an $\alpha > 0$. This, however, does not hold in the case $d \geq 2$ and $0 \neq V \in C_c^\infty$.

The aim of this paper is to provide a proof of Theorem (1) below. The proof uses the ideas of [3], but while there Zo's lemma plays a role and one needs something like $(*)$, here we make a better use of L^2-estimates and the Calderón–Zygmund decomposition, which allows us to dispose of the smoothness requirement on the kernels of the semigroup.

For $s \geq 0$ we define

$$\|f\|_{H(s)}^2 = \int |\widehat{f}(\omega)|^2 (1 + |\omega|^2)^s \, d\omega\,, \quad H(s) = \{f \in L^2 : \|f\|_{H(s)} < \infty\}\,,$$

where $\widehat{\ }$ is the Fourier transform.

(1) THEOREM. *If for some $\varepsilon > 0$, a non-zero $\varphi \in C_c^\infty(\mathbf{R}_+)$ and a constant C we have*

$$\|\varphi F_t\|_{H((d+1)/2+\varepsilon)} \leq C\,, \tag{2}$$

then $F(A)$ is of weak type $(1,1)$ and bounded on L^p for $1 < p < \infty$.

Proof. First observe that by (2), $\|F\|_{L^\infty} \leq C'C$, so

$$\|F(A)\|_{L^2,L^2} \leq C'C\,. \tag{3}$$

Consequently, by interpolation and duality, it is enough to prove that $F(A)$ is of weak type $(1,1)$. By the Trotter formula (see [4])

$$0 \leq e^{-tA}(x,y) \leq p_t(x-y)\,, \tag{4}$$

where $p_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/4t)$. Formula (4) implies the following inequalities:

$$\int e^{-tA}(x,y) e^{s|x-y|}\, dx \leq C e^{Cs^2 t}\,, \tag{5}$$

$$\int |e^{-tA}(x,y)|^2\, dx \leq C t^{-d/2}\,, \tag{6}$$

$$\sup_{x,y} |e^{-tA}(x,y)| \leq C t^{-d/2}\,, \tag{7}$$

for some constant C and all $s,t > 0$, $y \in \mathbf{R}^d$.

We write

$$\|K\|_a = \max\Big\{ \sup_x \int |K(x,y)|(1+|x-y|)^a\, dy,\ \sup_y \int |K(x,y)|(1+|x-y|)^a\, dx \Big\}\,.$$

(8) LEMMA. *If $\operatorname{supp} F \subset [-1,4]$, $\varepsilon > 0$, $a \geq 0$, then*

$$\|F(A)\|_a \leq C \|F\|_{H((d+1)/2+\varepsilon+a)}$$

where C is independent of F and A.

Proof. Put $K(\lambda) = F(-\log(\lambda))\lambda^{-1}$. We have

$$\|K\|_{H((d+1)/2+\varepsilon+a)} \leq C_1\|F\|_{h((d+1)/2+\varepsilon+a)}, \quad \operatorname{supp} K \subset [e^{-4}, e].$$

Let $K(\lambda) = \sum \widehat{K}(n)e^{in\lambda}$, $e_n = e^{ine^{-A}}e^{-A}$, whence $F(A) = K(e^{-A})e^{-A} = \sum \widehat{K}(n)e_n$. Inequalities (6) and (5) allow us to use (3.1) of [3] to obtain

$$\|e_n\|_a \leq C_2(1+|n|)^{d/2+a}$$

so

$$\begin{aligned}\|F\|_a &\leq C_2 \sum |\widehat{K}(n)|(1+|n|)^{d/2+a} \\ &\leq C_2\Big(\sum |\widehat{K}(n)|^2(1+|n|)^{d+2a+1+\varepsilon}\Big)^{1/2}\Big(\sum(1+|n|)^{-1-\varepsilon}\Big)^{1/2} \\ &\leq C_3\|K\|_{H((d+1+\varepsilon)/2+a)} \leq C_4\|F\|_{H((d+1)/2+\varepsilon+a)},\end{aligned}$$

which ends the proof.

(9) LEMMA. *For every $m \geq 0$ there exist $N, C > 0$ such that if $F \in H(N)$, $\operatorname{supp} F \subset [-1,4]$, then*

$$|F(A)(x,y)| \leq C\|F\|_{H(N)}(1+|x-y|)^{-m}$$

for all x, y, and A.

Proof. Put $G(\lambda) = F(\lambda)e^{\lambda}$, $N = d/2 + m + 1$. Of course $\|G\|_{H(N)} \leq C_1\|F\|_{H(N)}$. By Lemma (8), $\|G(A)\|_m \leq C_2\|G\|_{H(N)}$ and by (4) and (7),

$$\begin{aligned}|(1+|x-y|)^m F(A)(x,y)| &= \Big|\int G(A)(x,s)e^{-A}(s,y)(1+|x-y|)^m\,ds\Big| \\ &\leq \int |G(A)(x,s)|(1+|x-s|)^m e^{-A}(s,y)(1+|s-y|)^m\,ds \\ &\leq \|G(A)\|_m \sup_x p_1(x)(1+|x|)^m,\end{aligned}$$

and the lemma follows.

Of course (2) holds for every $\varphi \in C_c^\infty(\mathbf{R}_+)$. We fix φ and ψ such that φ, ψ are in $C^\infty(\mathbf{R})$, $\operatorname{supp}\varphi \subset [1/4, 2]$, $\sum \varphi(2^{2k}x) = 1$ for every $x > 0$, and $\operatorname{supp}\psi \subset [-1,1]$, with $\psi(x) = 1$ for $x \in [0, 1/2]$. Let

$$F_k(\lambda) = \varphi(2^{2k}\lambda)F(\lambda), \quad \psi_k(\lambda) = \psi(2^{2k}\lambda).$$

Choose $a < \varepsilon$. There exists C such that

$$\|(\psi_k F_k)(A)\|_{L^1,L^1} \leq C, \tag{10}$$

$$\int |F_k(A)|(x,y)(1+2^{-k}|x-y|)^a\,dx \leq C, \tag{11}$$

$$|\psi_k(A)|(x,y) \leq C2^{-kd}(1+2^{-k}|x-y|)^{-d-1}. \tag{12}$$

To see this we recall some simple properties of dilations. Let $\delta_t x = tx$ and $\delta_t f = f \circ \delta_t$. We have

$$\delta_{2^k} A \delta_{2^{-k}} = 2^{-2k}A_k \quad \text{where} \quad A_k = -\Delta + 2^{2k}V \circ \delta_{2^k}.$$

Hence

$$F_k(A) = \delta_{2^{-k}} \widetilde{F}_k(A_k)\delta_{2^k} \quad \text{where} \quad \widetilde{F}_k = F_k \circ \delta_{2^{-2k}} .$$

Replacing ε by $\varepsilon - a$ and applying (8) to $\widetilde{F}_k$ and A_k we obtain (11). The proof of (12) is similar but uses (9). (10) is a consequence of (11) and (12).

Let f be an integrable function. We fix the Calderón–Zygmund decomposition of f at height λ, that is, functions f_i and g and cubes Q_i such that

$$f = g + \sum f_i , \quad \operatorname{supp} f_i \subset Q_i , \quad \int |f_i| \le C\lambda |Q_i| ,$$
$$|g| \le C\lambda , \quad Q_i \cap Q_j = \emptyset \text{ for } i \ne j , \quad \sum |Q_i| \le C\|f\|_{L^1}/\lambda .$$

Let Q_i^* be the ball with the same center as Q_i and radius $2 \operatorname{diam} Q_i$. We put $k_i = [\log_2(\operatorname{diam} Q_i)]$. Let h be an integrable function such that $\operatorname{supp} h \subset \{x : |x| \le 1\} = B$. We have

$$\begin{aligned}
\int_{|x|>2} |F_k(A)h|(x)\,dx &\le \|h\|_{L^1} \sup_{y\in B} \int_{|x|>2} |F_k(A)|(x,y)\,dx \\
&\le \|h\|_{L^1} \sup_{y} \int_{|x-y|>1} |F_k(A)|(x,y)\,dx \\
&\le 2^{ka}\|h\|_{L^1} \sup_{y} \int |F_k(A)|(x,y)(1 + 2^{-k}|x-y|)^a\,dx \\
&\le C 2^{ka}\|h\|_{L^1}
\end{aligned}$$

and

$$\sum_{k\le 0} \int_{|x|>2} |F_k(A)h|(x)\,dx \le C \sum_{k\le 0} 2^{ka}\|h\|_{L^1} \le C_1\|h\|_{L^1} .$$

Using dilation we obtain

$$\sum_{j\le k_i} \int_{(Q_i^*)^c} |F_j(A)f_i|(x)\,dx \le C\|f_i\|_{L^1} . \tag{13}$$

(14) Lemma. *There exists C such that*

$$\Big\|\sum \psi_{k_i}(A)f_i\Big\|_{L^2}^2 \le C\lambda\|f\|_{L^1} .$$

Proof. First observe that there exists C_0 such that if $Q = \{x : \max|x_i| \le 1\}$ then for every x

$$\sup_{y\in Q}(1 + |x-y|)^{-d-1} \le C_0 \inf_{y\in Q}(1 + |x-y|)^{-d-1} .$$

Consequently, using dilations we obtain, for all i,

$$\sup_{y\in Q_i}(1 + 2^{-k_i}|x-y|)^{-d-1} \le C_0 \inf_{y\in Q_i}(1 + 2^{-k_i}|x-y|)^{-d-1} . \tag{15}$$

Fix i. Let y_0 be the center of Q_i. By (15)

$$\begin{aligned}|\psi_{k_i}(A)f_i|(x) &\le \int 2^{-k_i d}(1+2^{-k_i}|x-y|)^{-d-1}|f_i|(y)\,dy \\ &\le \lambda C_1|Q_i|2^{-k_i d}(1+2^{-k_i}|x-y_0|)^{-d-1} \\ &\le \lambda C_2 \int 2^{-k_i d}(1+2^{-k_i}|x-y|)^{-d-1}\chi_{Q_i}(y)\,dy \\ &= \lambda C_3(2^{-k_i d}(1+2^{-k_i}|\cdot|)^{-d-1} * \chi_{Q_i})(x).\end{aligned}$$

If $h \in L^2$, then

$$\begin{aligned}&|(h, 2^{-k_i d}(1+2^{-k_i}|\cdot|)^{-d-1} * \chi_{Q_i})| \\ &\qquad = |(2^{-k_i d}(1+2^{-k_i}|\cdot|)^{-d-1} * h, \chi_{Q_i})| \le C_4(Mh, \chi_{Q_i})\end{aligned}$$

where M is the Hardy–Littlewood maximal operator. The last inequality is the well-known property of the Hardy–Littlewood maximal operator (see for example Stein [5], Theorem 3.2). Since M is bounded on L^2,

$$\Big|\Big(h, \sum \psi_{k_i}(A)f_i\Big)\Big| \le C_5\Big(Mh, \sum \lambda\chi_{Q_i}\Big) \le C_6\|h\|_{L^2}\Big\|\sum \lambda\chi_{Q_i}\Big\|_{L^2}.$$

But $\|\sum \lambda\chi_{Q_i}\|_{L^2}^2 = \sum \lambda^2|Q_i| \le C\lambda\|f\|_{L^1}$, which ends the proof.

Clearly, if $j < k$, then $\psi_k F_j = 0$ so $\psi_k(A)F_j(A) = 0$. Similarly, if $j > k$ then $\psi_k(A)F_j(A) = F_j(A)$. Therefore

$$\begin{aligned}&F(A)f \\ &= \sum_{i,j} F_j(A)f_i + F(A)g \\ &= \sum_i\Big(\sum_{j\le k_i} F_j(A)f_i + \sum_{j>k_i} F_j(A)f_i\Big) + F(A)g \\ &= \sum_i\sum_{j\le k_i} F_j(A)f_i + \sum_{i,j} F_j(A)\psi_{k_i}(A)f_i - \sum_i F_{k_i}(A)\psi_{k_i}(A)f_i + F(A)g \\ &= \sum_i\sum_{j\le k_i} F_j(A)f_i + F(A)\Big(\sum \psi_{k_i}(A)f_i + g\Big) - \sum_i F_{k_i}(A)\psi_{k_i}(A)f_i.\end{aligned}$$

Putting $S = \bigcup Q_i^*$, by (13) and the properties of the Calderón–Zygmund decomposition we have

$$\begin{aligned}\Big|\Big\{x : \Big|\sum_i\sum_{j\le k_i} F_j(A)f_i\Big| > \lambda/3\Big\}\Big| &\le |S| + (3/\lambda)\int_{S^c}\Big|\sum_i\sum_{j\le k_i} F_j(A)f_i\Big| \\ &\le C\|f\|_{L^1}/\lambda + (C/\lambda)\sum\|f_i\|_{L^1} \\ &\le C\|f\|_{L^1}/\lambda.\end{aligned}$$

Next, by (14),

$$\Big\| \sum \psi_{k_i}(A)f_i + g \Big\|_{L^2}^2 \le C\lambda \|f\|_{L^1},$$

and by (3)

$$\begin{aligned} \Big| \Big\{ x : \Big| F(A)\Big(\sum \psi_{k_i}(A)f_i + g\Big)\Big| > \lambda/3 \Big\} \Big| & \\ \le (C/\lambda^2)\Big\| \sum \psi_{k_i}(A)f_i + g \Big\|_{L^2}^2 & \\ \le C'\lambda \|f\|_{L^1}/\lambda^2 = C\|f\|_{L^1}/\lambda . & \end{aligned}$$

Finally, by (10),

$$\begin{aligned} \Big| \Big\{ x : \Big| \sum F_{k_i}(A)\psi_{k_i}(A)f_i \Big| > \lambda/3 \Big\} \Big| &\le 3 \Big\| \sum F_{k_i}(A)\psi_{k_i}(A)f_i \Big\|_{L^1} / \lambda \\ &\le (C/\lambda) \sum \|f_i\|_{L^1} \le C\|f\|_{L^1} \le C\|f\|_{L^1}/\lambda , \end{aligned}$$

which ends the proof of (1).

REFERENCES

[1] M. Christ, *A Hörmander multiplier theorem on nilpotent groups*, preprint.

[2] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.

[3] W. Hebisch, *Almost everywhere summability of eigenfunction expansions associated to elliptic operators*, Studia Math. 96 (1990).

[4] T. Kato, *Trotter's formula for an arbitrary pair of selfadjoint contraction semigroups*, in: Topics in Functional Analysis, I. Gohberg and M. Kac (eds.), Adv. in Math. Suppl. Stud. 3, Academic Press, New York 1978, 185–195.

[5] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

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