COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 1

ON LEVEL SETS OF APPROXIMATE DERIVATIVE AND APPROXIMATE SYMMETRIC DERIVATIVE

BY

H. H. PU AND H. W. PU (COLLEGE STATION, TEXAS)

Let f be a real-valued function on the real line R, I be a closed interval, and E be the set of points where f is differentiable. In [2], O'Malley proved that if f is approximately differentiable at every $x \in I$, then

$$\{x \in I: f'_{\mathrm{ap}}(x) = \lambda\} \neq \emptyset$$

implies

$$\{x \in I: f'_{\mathrm{ap}}(x) = \lambda\} \cap E \neq \emptyset.$$

This result was extended in [3] to functions having approximate derivative (finite or infinite) and satisfying a certain condition, which is weaker than approximate continuity, on I. The purpose of the present paper is two-fold: to examine firstly whether the above conclusion is still true if a countable exceptional set is allowed for the existence of approximate derivative; and, secondly, whether there is an analogous theorem for approximate symmetric derivative $f_{\mathrm{ap}}^{(\prime)}$ and symmetric derivative $f^{(\prime)}$. The definitions of these derivatives can be found in [1]. The answers to the questions are negative even if the function f is continuous.

Before we proceed, we mention that there exists a sequence of intervals, with positive endpoints converging to 0, whose union has 0 as a point of dispersion. For example,

$$I_n = [2^{-n}, 2^{-n}(n+3)/(n+2)], \qquad n = 1, 2, \ldots$$

The set

$$\bigcup_{n=1}^{\infty} [2^{-n}(n+3)/(n+2), 2^{-n+1}],$$

of course, has 0 as a point of right density.

THEOREM 1. *There exists a continuous function f which is approximately differentiable nearly everywhere and $f'_{\mathrm{ap}}(0) = 0$, but*

$$\{x: f'_{\mathrm{ap}}(x) = 0\} \cap E = \emptyset.$$

Proof. Let $\{x_n\}$ be a strictly decreasing sequence of positive numbers converging to 0 such that $\bigcup_{n=1}^{\infty} [x_{2n+1}, x_{2n}]$ has 0 as a point of right density. We define f as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \in \{0\} \cup \bigcup_{n=1}^{\infty} [x_{2n+1}, x_{2n}] \cup \{x\colon x \geqslant 1\}, \\ x & \text{if } x = \tfrac{1}{2}(x_{2n}+x_{2n-1}) \text{ for } n = 1, 2, \ldots, \\ \text{linear on } [x_{2n}, \tfrac{1}{2}(x_{2n}+x_{2n-1})] \text{ and on} & \\ \qquad [\tfrac{1}{2}(x_{2n}+x_{2n-1}), x_{2n-1}] \text{ for } n = 1, 2, \ldots, & \\ f(-x) & \text{if } x < 0. \end{cases}$$

Then

$$f'_{\mathrm{ap}}(0) = \lim_{h \to 0} \mathrm{ap} \frac{f(h)-0}{h} = \lim_{\substack{h \in A \\ h \to 0}} \frac{f(h)}{h} = 0,$$

where

$$A = \bigcup_{n=1}^{\infty} [x_{2n+1}, x_{2n}] \cup \bigcup_{n=1}^{\infty} [-x_{2n}, -x_{2n+1}].$$

However,

$$\tfrac{1}{2}(x_{2n}+x_{2n-1}) \to 0 \quad \text{and} \quad f\left(\tfrac{1}{2}(x_{2n}+x_{2n-1})\right)/\tfrac{1}{2}(x_{2n}+x_{2n-1}) \to 1$$

as $n \to \infty$. This shows that $f'(0)$ does not exist.

Clearly,

$$E = R - \{0\} - \{x\colon x = \pm x_n \text{ or } \pm\tfrac{1}{2}(x_{2n}+x_{2n-1}),\ n = 1, 2, \ldots\}$$

and $f'(x)$ is either equal to $2x$ or the slope of some linear segment of the graph of f. It follows that $f'(x) \neq 0$ for any $x \in E$. The theorem is proved.

THEOREM 2. *There exists a continuous function f which has a finite approximate symmetric derivative $f^{(s)}_{\mathrm{ap}}(x)$ at every x and $f^{(s)}_{\mathrm{ap}}(0) = 0$. Also, $f^{(s)}_{\mathrm{ap}}$ is a Baire 1 function with Darboux property. However,*

$$\{x\colon f^{(s)}_{\mathrm{ap}}(x) = 0\} \cap E^{(s)} = \emptyset,$$

$E^{(s)}$ being the set of points x where f is symmetrically derivable.

Proof. Let $x_0 = 1$ and, for each positive integer n,

$$x_n = 2^{-n}, \quad x'_n = 2^{-n}(n+3)/(n+2), \quad \xi_n = 2^{-n}(3n+4)/(2n+3).$$

Thus $x_n < x'_n < \xi_n < x_{n-1}$ for each n. Let

$$z_n = \tfrac{1}{2}(x_n+x'_n), \quad z'_n = \tfrac{1}{2}(z_n+x'_n) \quad \text{and} \quad z''_n = \tfrac{1}{2}(x_n+z_n).$$

In other words,

$$z_n = 2^{-n}(2n+5)/(2n+4), \quad z'_n = 2^{-n}(4n+11)/(4n+8)$$

and

$$z''_n = 2^{-n}(4n+9)/(4n+8).$$

We define two functions φ_1 and ψ_1 on $[0, 1]$ as follows:

$$\varphi_1(x) = \begin{cases} \frac{3}{2}x & \text{if } x = 0 \text{ or } x = x_n,\ n = 0, 1, 2, \ldots, \\ 2x & \text{if } x \in [z_n', \xi_n],\ n = 1, 2, \ldots, \\ \text{linear on } [x_n, z_n'] \text{ and on } [\xi_n, x_{n-1}], & n = 1, 2, \ldots; \end{cases}$$

$$\psi_1(x) = \begin{cases} \frac{3}{2}x & \text{if } x = 0 \text{ or } x = x_n,\ n = 0, 1, 2, \ldots, \\ 2x & \text{if } x \in [z_n'', \xi_n],\ n = 1, 2, \ldots, \\ \text{linear on } [x_n, z_n''] \text{ and on } [\xi_n, x_{n-1}], & n = 1, 2, \ldots \end{cases}$$

Clearly, $\varphi_1 = \psi_1$ on $[z_n', x_{n-1}]$ and, by simple calculation, their slope on (ξ_n, x_{n-1}) is

$$\frac{(3/2)x_{n-1} - 2\xi_n}{x_{n-1} - \xi_n} = \frac{1}{n+2}.$$

The slope of φ_1 on (x_n, z_n') is

$$\frac{2z_n' - (3/2)x_n}{z_n' - x_n} = \frac{2n+10}{3}$$

and that of ψ_1 on (x_n, z_n'') is

$$\frac{2z_n'' - (3/2)x_n}{z_n'' - x_n} = 2n+6.$$

Next we round off the "corners" of φ_1 and ψ_1 to obtain functions φ and ψ on $[0, 1]$, respectively, so that the values at z_n are not changed, $\varphi(x) = \psi(x)$ on $[x_n', x_{n-1}]$ for $n = 1, 2, \ldots$, and φ and ψ are differentiable with $\varphi'(x) > 0$, $\psi'(x) > 0$ at every $x \in (0, 1]$.

Now we define f by

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \in [0, 1], \\ \psi(-x) & \text{if } x \in [-1, 0), \\ x/3 & \text{if } x > 1, \\ -x/3 & \text{if } x < -1. \end{cases}$$

Then f is continuous on R and differentiable at every $x \neq 0$. At $x = 0$, since $\varphi = \psi$ on

$$A = \bigcup_{n=1}^{\infty} (x_n', x_{n-1})$$

which has 0 as a point of right density, $f(0+h) - f(0-h) = 0$ for every $h \in A$. It follows that $f_{\text{ap}}^{(s)}(0) = 0$.

For each n, $z_n \in (x_n, z_n')$ on which $\varphi_1' = (2n+10)/3$ and $z_n \in (z_n'', \xi_n)$ on which $\psi_1' = 2$. Therefore we have

$$f(z_n) = \varphi(z_n) = \varphi_1(z_n) = \varphi_1(x_n) + \frac{2n+10}{3}(z_n - x_n) = \frac{11n+28}{6n+12}\frac{1}{2^n},$$

$$f(-z_n) = \psi(z_n) = \psi_1(z_n) = \psi_1(z_n'') + 2(z_n - z_n'') = \frac{2n+5}{n+2}\frac{1}{2^n},$$

and

$$\lim_{n\to\infty} \frac{f(z_n)-f(-z_n)}{2z_n} = \lim_{n\to\infty} \frac{-n-2}{12n+30} = -\frac{1}{12} \neq 0 = f_{ap}^{(\prime)}(0).$$

Since $z_n \to 0$ as $n \to \infty$, we see that $f^{(\prime)}(0)$ does not exist.

Clearly, $E^{(\prime)} = R - \{0\}$. In fact, we have $f^{(\prime)}(x) = f'(x) > 0$ at every $x > 0$ and $f^{(\prime)}(x) = f'(x) < 0$ at every $x < 0$. Consequently,

$$\{x\colon f_{ap}^{(\prime)}(x) = 0\} \cap E^{(\prime)} = \emptyset.$$

To see that $f_{ap}^{(\prime)}$ is Baire 1, we need only to observe that

$$f_{ap}^{(\prime)}|(R - \{0\}) = f'$$

which is Baire 1 on $R-\{0\}$. Finally, let t_n be any point in (ξ_n, x_{n-1}) such that $\varphi'(t_n) = \varphi_1'(t_n)$ for each n. Then the sequence $\{t_n\}$ decreases to 0 and

$$\lim_{n\to\infty} f_{ap}^{(\prime)}(t_n) = \lim_{n\to\infty} \varphi_1'(t_n) = \lim_{n\to\infty} \frac{1}{n+2} = 0,$$

while the sequence $\{-t_n\}$ increases to 0 and

$$\lim_{n\to\infty} f_{ap}^{(\prime)}(-t_n) = \lim_{n\to\infty} \left(-\psi_1'(t_n)\right) = -\lim_{n\to\infty} \frac{1}{n+2} = 0.$$

By a theorem of Young [4], $f_{ap}^{(\prime)}$ has Darboux property. The proof is completed.

REFERENCES

[1] S. N. Mukhopadhyay, *On approximate Schwarz differentiability*, Monatsh. Math. 70 (1966), pp. 454–460.

[2] R. J. O'Malley, *The set where an approximate derivative is a derivative*, Proc. Amer. Math. Soc. 54 (1976), pp. 122–124.

[3] H. H. Pu and H. W. Pu, *On the set where an approximate derivative is a derivative*, Colloq. Math. 43 (1980), pp. 155–160.

[4] J. Young, *A theorem in the theory of functions of a real variable*, Rend. Circ. Mat. Palermo 24 (1907), pp. 187–192.

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TEXAS

Reçu par la Rédaction le 16.6.1983