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ON THE SEPARATION PROPERTIES OF K_ω

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1. Introduction. We shall consider the transfinite extension of the compactness degree of a space, introduced by J. de Groot (see [4]), which we shall call the transfinite compactness degree (see Defintion 2.1). The main goal of this note is to show that for the space K_ω consisting of the points of the Hilbert cube with at most finitely many non-zero coordinates the compactness degree is not defined. This extends the well-known fact that for K_ω the small transfinite inductive dimension is not defined (see [3]). We shall also prove another separation property of K_ω, that the intersection of finitely many partitions between distinct pairs of the opposite faces of K_ω is not locally compact. Finally, an example is given, using K_ω, of a perfect map with finite-dimensional fibers of a space for which the transfinite compactness degree is not defined on a zero-dimensional space.

2. Terminology and notation. All spaces which we consider in this note are assumed to be metrizable and separable. Our topological terminology will follow [1] and [2].

Throughout the paper ω denotes the set of natural numbers, I – the real unit interval, I^ω – the Hilbert cube and let $K_\omega = \{x \in I^\omega$: x has only finitely many non-zero coordinates$\}$, p_i: $K_\omega \to I$ be the projection of K_ω onto i'th axis and $p_i^{-1}(1)$, $p_i^{-1}(0)$ be the pair of i'th opposite faces in K_ω.

The following notion of transfinite compactness degree extends the compactness degree defined by de Groot [4] in the same way as transfinite inductive dimension extends the classical inductive dimension ind, see [3]:

Definition 2.1. The *transfinite compactness degree* $\operatorname{cmp} X$ of a space X is defined as follows:

(i) $\operatorname{cmp} X = -1$ if and only if X is compact;

(ii) $\operatorname{cmp} X \leqslant \alpha$, where α is an ordinal number, if for every point $x \in X$ and each neighborhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{cmp} \operatorname{Fr} U < \alpha$ ($\operatorname{Fr} U$ – the boundary of U);

(iii) $\operatorname{cmp} X = \alpha$ if $\operatorname{cmp} X \leqslant \alpha$ and the inequality $\operatorname{cmp} X < \alpha$ does not hold.

If there is no ordinal α with $\operatorname{cmp} X = \alpha$ we say that X *has no transfinite compactness degree.*

3. Results. The main result of this note is the following

THEOREM 3.1. *The space K_ω has no transfinite compactness degree.*

A result closely related to Theorem 3.1 is the following

Example 3.2. There exists a map $f\colon X \to Y$ which is perfect (i.e. closed and with compact fibers), has finite-dimensional fibers and maps the space X for which the transfinite compactness degree is not defined onto a zero-dimensional space Y.

Finally, we shall prove the following simple fact about the separation properties in K_ω:

PROPOSITION 3.3. *Let L_i be a partition in K_ω between the i'th opposite faces* (see Section 2). *Then no finite intersection $L_1 \cap \ldots \cap L_k$ is locally compact.*

4. The proof of Theorem 3.1. In the proof we shall use the following modification of the notion of transfinite compactness degree:

Definition 4.1. Let α be either an ordinal number $= 0$ or the integer -1; then we define

(i) $\operatorname{cmp}_k X = -1$ if and only if X is compact;

(ii) $\operatorname{cmp}_k X \leqslant \alpha$ if for every pair A, B of disjoint compact subsets of X there exists a partition L between A and B such that $\operatorname{cmp}_k L < \alpha$;

(iii) $\operatorname{cmp}_k X = \alpha$ if $\operatorname{cmp}_k X \leqslant \alpha$ and the inequality $\operatorname{cmp}_k X < \alpha$ does not hold.

It is easy to see that cmp_k is monotone with respect to closed subspaces (cf. [4], Th. 3.2.1), i.e.

$$(*) \qquad \operatorname{cmp}_k A \leqslant \operatorname{cmp}_k X \quad \text{if } A \subset X \text{ is closed.}$$

LEMMA 4.2. *Let X_1 and X_2 be closed subsets of a metrizable space $X = X_1 \cup X_2$; then*

$$\operatorname{cmp}_k X \leqslant \operatorname{cmp}_k X_1 + \operatorname{cmp}_k X_2 + 1.$$

Proof. The proof of Lemma 4.2 is by transfinite induction on $k(X_1, X_2) = \operatorname{cmp}_k X_1 + \operatorname{cmp}_k X_2$. The case $k(X_1, X_2) = -2$ is trivial. Assume that we have verified the assertion for $k(X_1, X_2) < \alpha$. Let $\operatorname{cmp}_k X_1 = \alpha_1$, $\operatorname{cmp}_k X_2 = \alpha_2$, where $\alpha_1 + \alpha_2 = \alpha$. Consider two disjoint compact subsets A and B of X. Our lemma will be proved if we find a partition L between A and B such that $\operatorname{cmp}_k L \leqslant \alpha_1 + \alpha_2$. Since $\operatorname{cmp}_k X_2 = \alpha_2$, there exists a partition L' in X between $X_2 \cap A$ and $X_2 \cap B$ such that $\operatorname{cmp}_k L' = \beta < \alpha_2$. Let L be a partition in X between A and B which extends L'. By $(*)$ $\operatorname{cmp}_k (L \cap X_1) \leqslant \alpha_1$.

We have

$$\operatorname{cmp}_k L = \operatorname{cmp}_k((L \cap X_1) \cup L') \leqslant \alpha_1 + \beta + 1 \leqslant \alpha_1 + \alpha_2.$$

Thus the proof is completed.

LEMMA 4.3. *If X has transfinite compactness degree then the ordinal $\operatorname{cmp}_k X$ is also defined.*

Proof. Let X has transfinite compactness degree.

It is enough to prove that if A and B are disjoint compact subsets of X then there exists a partition L between A and B such that L has cmp_k. This can be shown by transfinite induction on $\operatorname{cmp} X$. If $\operatorname{cmp} X = -1$ then the proposition is true. Assume that the proposition is true for all ordinal numbers less than α. For every point $x \in A$ there exists an open neighborhood U_x of the point x such that $U_x \cap B = \emptyset$ and $\operatorname{cmp} \operatorname{Fr} U_x < \alpha$. It follows by our assumption that $\operatorname{Fr} U_x$ has cmp_k. Since A is compact we can choose a finite number of points $x_1, x_2, \ldots, x_k \in A$ such that $A \subset U_{x_1} \cup \ldots \cup U_{x_k}$. Let $U = \bigcup_{i=1}^{k} U_{x_i}$, then $L = \operatorname{Fr} U$ is a partition between A and B and $\operatorname{cmp}_k L = \operatorname{cmp}_k \operatorname{Fr} U \leqslant \operatorname{cmp}_k \bigcup_{i=1}^{k} \operatorname{Fr} U_{x_i}$. By virtue of Lemma 4.2 L has cmp_k.

Now we are in position to prove Theorem 3.1.

According to Lemma 4.3 it is sufficient to show that K_ω has no cmp_k. Conversely, assume that K_ω has cmp_k and let

$$\alpha = \min\{\operatorname{cmp} L\colon\ L = L_1 \cap \ldots \cap L_k,\ k = 1, 2, \ldots\},$$

where L_i is a partition between the pair of ith opposite faces in K_ω (see Section 2).

First let us check that $\alpha \neq -1$, i.e. that $L_1 \cap \ldots \cap L_m$ is not compact for $m = 1, 2, \ldots$ Indeed, since in I^n the intersection of partitions between distinct pairs of the opposite faces of I^n is not empty, for $n = 1, 2, \ldots$ (cf. [2], Th. 1.8.1) we can choose

$$a_i \in (L_1 \cap \ldots \cap L_m) \cap (I^m \times \underbrace{\{1\} \times \ldots \times \{1\}}_{i} \times \{0\} \times \ldots) \quad \text{for } i = 1, 2, \ldots$$

The sequence $\{a_i\}_{i=1}^{\infty}$ has no limit points in K_ω, thus $L_1 \cap \ldots \cap L_m$ is not compact.

Now we will show that if $\operatorname{cmp}_k(L_1 \cap \ldots \cap L_m) = \alpha$ then we can find other partitions $L'_1, L'_2, \ldots, L'_{m+1}$, where L'_i is a partition between the pair of i'th opposite faces in K_ω such that $\operatorname{cmp}_k(L'_1 \cap \ldots \cap L'_{m+1}) < \alpha$. This will contradict the choice of α. Let

$$A_{m+1} = (I^{m+1} \times \{0\} \times \{0\} \times \ldots) \cap p_{m+1}^{-1}(0),$$
$$B_{m+1} = (I^{m+1} \times \{0\} \times \{0\} \times \ldots) \cap p_{m+1}^{-1}(1).$$

Since these sets are disjoint and compact there exists a partition L' between $A_{m+1} \cap L_1 \cap \ldots \cap L_m$ and $B_{m+1} \cap L_1 \cap \ldots \cap L_m$ in $L_1 \cap \ldots \cap L_m$ such that $\mathrm{cmp}_k(L' \cap L_1 \cap \ldots \cap L_m) < \alpha$. Let us extend L' to a partition L in K_ω between A_{m+1} and B_{m+1}. Clearly $\mathrm{cmp}_k(L \cap L_1 \cap \ldots \cap L_m) < \alpha$. Let

$$X = (L \cap p_{m+1}^{-1}(1)) \cup (L \cap p_{m+1}^{-1}(0)).$$

If $X = \emptyset$ then L is a partition between $p_{m+1}^{-1}(0)$ and $p_{m+1}^{-1}(1)$. Putting $L'_{m+1} = L$ and $L'_i = L$ for $i = 1, 2, \ldots, n$ we get required partitions. Assume $X \neq \emptyset$. The set X is closed in K_ω and disjoint from $I^{m+1} \times \{0\} \times \{0\} \times \ldots$, so we can find an open set U in I^ω such that:

$$I^{m+1} \times \{0\} \times \{0\} \times \ldots \subset U \cap K_\omega \subset \bar{U} \cap K_\omega \subset K_\omega \setminus X.$$

By the Wallace Lemma (see [1]) there exist positive numbers $c_1, c_2, \ldots, c_p$ such that

$$I^{m+1} \times \{0\} \times \{0\} \times \ldots \subset ([0, c_1] \times \ldots \times [0, c_p] \times I \times I \times \ldots) \cap K_\omega \subset K_\omega \setminus X.$$

Let $K = ([0, c_1] \times \ldots \times [0, c_p] \times I \times I \times \ldots) \cap K_\omega$, then

$$L \cap K \cap p_{m+1}^{-1}(1) \subset L \cap U \cap K_\omega \cap p_{m+1}^{-1}(1) = \emptyset;$$

the proof that $L \cap K \cap p_{m+1}^{-1}(0) = \emptyset$ is similar. Therefore $M_{m+1} = K \cap L$ is a partition between $p_{m+1}^{-1}(1) \cap K$ and $p_{m+1}^{-1}(0) \cap K$. Obviously $M_i = L_i \cap K$ is a partition between $p_i^{-1}(0) \cap K$ and $p_i^{-1}(1) \cap K$ for $i = 1, 2, \ldots, m$. By (*) it follows that

$$\mathrm{cmp}_k(M_1 \cap \ldots \cap M_{m+1}) = \mathrm{cmp}_k(K \cap L_1 \cap \ldots \cap L_m \cap L) < \alpha.$$

Now, let $f\colon K \to K_\omega$ be a homeomorphism defined by the formula:

$$f((x_1, x_2, \ldots)) = \left(\frac{x_1}{c_1}, \frac{x_2}{c_2}, \ldots, \frac{x_p}{c_p}, x_{p+1}, \ldots\right).$$

Let us put $f(M_i) = L'_i$ for $i = 1, 2, \ldots, m+1$. It is easy to check that $L'_1, \ldots, L'_{m+1}$ are required partitions. This completes the proof.

5. An example. Let $J = \{0\} \cup [\frac{1}{2}, 1] \subset I$. By a minor modification of the reasoning in Section 4 one can prove that the space $J^\omega \cap K_\omega$ has no transfinite compactness degree. More specifically, one has to consider "the i'th opposite faces" $p_i^{-1}(1)$ and $p_i^{-1}(\frac{1}{2})$ in $J^\omega \cap K_\omega$ ($p_i\colon J^\omega \cap K_\omega \to J$ being the i'th projection) and then the arguments given in Section 4 lead to a contradiction with the existence of the number

$$\alpha = \min \{\mathrm{cmp}_k(L_1 \cap \ldots \cap L_k)\colon k = 1, 2, \ldots\},$$

where L_i is a partition between $p_i^{-1}(1)$ and $p_i^{-1}(\frac{1}{2})$ in $J^\omega \cap K_\omega$.

Now let $X = J^\omega \cap K_\omega$, $Y = \{0, 1\}^\omega \cap K_\omega$ and let $f\colon X \to Y$ be defined by the formula:

$$f((x_i)) = (y_i) \quad \text{where} \quad y_i = \begin{cases} 0 & \text{if} \quad x_i = 0, \\ 1 & \text{if} \quad x_i \in [\frac{1}{2}, 1]. \end{cases}$$

The function f is perfect since it is closed and the inverse image of each $y \in Y$ is homeomorphic to I^n for some $n \in \omega$. Moreover it is easy to check that f is open.

6. The proof of Proposition 3.3. Let us assume that there exists $k \in \omega$ such that $L_1 \cap \ldots \cap L_k$ is locally compact. Then for every $x \in (I^k \times \{0\} \times \times \{0\} \times \ldots) \cap L_1 \cap \ldots \cap L_k$ there exists an open neighborhood U_x of x in K_ω such that $\bar{U}_x$ in $L_1 \cap \ldots \cap L_k$ is compact. Since $I^k \times \{0\} \times \ldots$ is compact we can choose $x_1, \ldots, x_m$ such that $(I^k \times \{0\} \times \ldots) \cap L_1 \cap \ldots \cap L_k \subset U_{x_1} \cap \ldots \cap U_{x_m}$. For every $y \in (I^k \times \{0\} \times \ldots) \setminus L_1 \cap \ldots \cap L_k$ there exists an open neighborhood V_y of y in K_ω such that

$$V_y \cap L_1 \cap \ldots \cap L_k = \emptyset \quad \text{and} \quad I^k \times \{0\} \times \ldots \subset \bigcup V_y \cup \bigcup_{i=1}^{m} U_{x_i}.$$

Proceeding in the same way as in Theorem 3.1 we can find $K \subset K_\omega$ homeomorphic to K_ω such that

$$I^k \times \{0\} \times \ldots \subset K \subset \bigcup V_y \cup \bigcup_{i=1}^{m} U_{x_i}.$$

Then

$$K \cap L_1 \cap \ldots \cap L_k \subset \Big(\bigcup_{i=1}^{m} U_{x_i}\Big) \cap L_1 \cap \ldots \cap L_k,$$

and thus the intersection being a closed subset of the compact set $\big(\bigcup_{i=1}^{m} U_{x_i}\big) \cap L_1 \cap \ldots \cap L_k$ is compact. But we have shown in Theorem 3.1 that the intersection of finite partitions between the pairs of i'th opposite faces in K_ω cannot be compact.

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