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$\mathscr{F}$-SINGULAR AND $\mathscr{G}$-COSINGULAR OPERATORS

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In this paper we generalize strictly singular and strictly cosingular operators as given in [7]. This generalization will include not only strictly singular and strictly cosingular operators, but also unconditionally converging operators studied by Pełczyński in [6] and almost weakly compact operators studied by Herman in [4].

We intend to preserve the notation of [2]. All spaces are to be Banach spaces and all maps are to be continuous. If X and Y are Banach spaces, then the conjugate spaces are denoted by X' and Y' respectively. Also T' is the adjoint operator to T. By a class of Banach spaces we mean Banach spaces with a given property.

Definition 0.1. Let $\mathscr{F}$ be a class of Banach spaces. An operator $T: X \to Y$ is said to be *$\mathscr{F}$-singular* provided that for no Banach space F in $\mathscr{F}$ does there exist isomorphic embeddings $i_1: F \to X$ and $i_2: F \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] & & Y \\ & F \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

is commutative.

Definition 0.2. Let $\mathscr{G}$ be a class of Banach spaces. An operator $T: X \to Y$ is said to be *$\mathscr{G}$-cosingular* provided that for no Banach space G in $\mathscr{G}$ does there exist epimorphisms $h_1: X \to G$ and $h_2: Y \to G$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "h_1"'] & & Y \arrow[dl, "h_2"] \\ & G & \end{tikzcd}$$

is commutative.

If $\mathscr{F}$ (resp., $\mathscr{G}$) consists of only one Banach space E, then we shall say that T is *E-singular* (resp. *E-cosingular*).

We recall that a linear operator $T\colon X \to Y$ is *weakly compact* if it maps bounded sets in X into weakly sequentially compact sets. $T\colon X \to Y$ is almost weakly compact if, whenever T has a bounded inverse on a closed subspace M of X, then M is reflexive. Note that every weakly compact operator is almost weakly compact. A linear operator $T\colon X \to Y$ is said to be *unconditionally converging* (uc operator) if it sends every weakly unconditionally converging series in X into an unconditionally converging series in Y. In [5] it is shown that $T\colon X \to Y$ is a uc operator if and only if T has no bounded inverse on a subspace E of X isomorphic to c_0.

1. $\mathscr{F}$-singular operators. Properties for $\mathscr{F}$-singular operators are now given.

PROPOSITION 1.1. *Let $\mathscr{F}$ and $\mathscr{F}'$ be classes of Banach spaces. Suppose for every F in $\mathscr{F}$ there exists an E in $\mathscr{F}'$ such that E is isomorphic (linearly homeomorphic) to F, and conversely, for every E in $\mathscr{F}'$ there exists an F in $\mathscr{F}$ such that F is isomorphic to E. Then T is $\mathscr{F}$-singular if and only if T is $\mathscr{F}'$-singular.*

The proof is clear.

Remark 1. An equivalent condition is:

A linear operator $T\colon X \to Y$ is $\mathscr{F}$-singular if whenever T has a bounded inverse on M, a closed subspace of X, M does not belong to $\mathscr{F}$.

Remark 2. (a) If $\mathscr{F}$ is the class of all infinite-dimensional Banach spaces, T is $\mathscr{F}$-singular if and only if T is strictly singular (see [7]).

(b) If $\mathscr{F}$ is the class of all non-reflexive Banach spaces, T is $\mathscr{F}$-singular if and only if T is almost weakly compact.

(c) If $\mathscr{F}$ is the class of all Banach spaces isomorphic to c_0, T is $\mathscr{F}$-singular if and only if T is a uc operator.

Remark 3. If $\mathscr{F}_1 \subseteq \mathscr{F}_2$, where $\mathscr{F}_1$ and $\mathscr{F}_2$ are classes of Banach spaces, then T $\mathscr{F}_2$-singular implies T $\mathscr{F}_1$-singular. Hence if T is strictly singular, T is almost weakly compact and if T is almost weakly compact, T is a uc operator.

PROPOSITION 1.2. *The set of $\mathscr{F}$-singular operators is closed in the uniform operator topology of $L(X, Y)$.*

The proof is similar to that given for theorem III. 2.4 in [3].

PROPOSITION 1.3. *Let $T \in L(X, Y)$ be $\mathscr{F}$-singular; and let $0 \neq R \in L(X, Z)$ and $0 \neq S \in L(V, X)$. Then RT and TS are $\mathscr{F}$-singular.*

Proof. Suppose RT has a bounded inverse on a subspace N of X. Then there exists a $c > 0$ such that, for all x in N,

$$\|R\|\,\|T(x)\| \geqslant \|RT(x)\| \geqslant c\|x\|.$$

Thus T has a bounded inverse on N, whence N does not belong to $\mathscr{F}$. Therefore, RT is $\mathscr{F}$-singular.

Suppose TS has a bounded inverse on a subspace M of V. Then there exists a $c > 0$ such that, for all x in M,

$$\|TS(x)\| \geqslant c\|x\| \geqslant \frac{c}{\|S\|} \cdot \|S(x)\|.$$

Thus T has a bounded inverse on $S(M)$, and therefore $S(M)$ does not belong to $\mathscr{F}$. But S is 1-1 on M since TS is 1-1 on M, and since $S^{-1} = (TS)^{-1} \cdot T$, S^{-1} is continuous on $S(M)$. So $S(M)$ is isomorphic to M, and by proposition 1.1, TS is $\mathscr{F}$-singular.

Example 1.4. *If T and S are $\mathscr{F}$-singular, then $S+T$ is not necessarily $\mathscr{F}$-singular.*

Proof. Let $\mathscr{F}$ be the class of all Banach spaces isomorphic to $J \times J$, where J denotes the James space (see [1]). Let S and T be the natural projections of $J \times J$ onto J regarded as subspaces $J \times 0$ and $0 \times J$ of J. Then $S+T$ is the identity operator, hence it is not $(J \times J)$-singular. Now from section 2 of [1] there cannot be an isomorphic embedding from $J \times J$ into J such that the diagram is commutative. This implies S and are $(J \times J)$-singular.

2. $\mathscr{G}$-cosingular operators. If $\mathscr{G}_1 \subseteq \mathscr{G}_2$, where $\mathscr{G}_1$ and $\mathscr{G}_2$ are classes of Banach spaces, then T $\mathscr{G}_2$-cosingular implies T $\mathscr{G}_1$-cosingular. If $\mathscr{G}$ if the class of all infinite-dimensional Banach spaces, T is $\mathscr{G}$-cosingular is and only if T is strictly cosingular (see [7]). If $\mathscr{G}$ is the class of all Banach spaces isomorphic to M, we call T an *M-cosingular operator.*

PROPOSITION 2.1. *The set of $\mathscr{G}$-cosingular operators is closed in the uniform operator topology of $L(X, Y)$.*

The proof is similar to that given for proposition 1 of [7].

PROPOSITION 2.2. *Let $T \in L(X, Y)$ be $\mathscr{G}$-cosingular; and let $0 \neq R \in L(Y, Z)$ and $0 \neq S \in L(V, X)$. Then RT and TS are $\mathscr{G}$-cosingular.*

The proof is clear.

Example 2.3. *If T and S are $\mathscr{G}$-cosingular, then $S+T$ is not necessarily $\mathscr{G}$-cosingular.*

Proof. S and T defined as in example 1.4 will be $(J \times J)$-cosingular but $S+T$ will not be.

3. Duals. Dual relations between singular and cosingular are now considered.

PROPOSITION 3.1. *Let $T\colon X \to Y$ and let $\mathscr{F}$ and $\mathscr{G}$ be classes of Banach spaces.*

(a) *If the dual of every G in $\mathscr{G}$ contains a Banach space F in $\mathscr{F}$, and if T' is $\mathscr{F}$-singular, then T is $\mathscr{G}$-cosingular.*

(b) *If the dual of every F in $\mathscr{F}$ contains a Banach space G in $\mathscr{G}$, and if T' is $\mathscr{G}$-cosingular, then T is $\mathscr{F}$-singular.*

The proof is based on the fact the operator adjoint to an epimorphism (isomorphic embedding) is an isomorphic embedding (epimorphism). (See theorems II. 3.11 and II. 3.13 of [3].)

COROLLARY 3.2. *Let $T: X \to Y$.*

(a) *If T'' is $\mathscr{F}$-singular, then T is $\mathscr{F}$-singular.*

(b) *If T'' is $\mathscr{G}$-cosingular, then T is $\mathscr{G}$-cosingular.*

Remark 4. The converses to proposition 3.1 (and corollary 3.2) are not true since, in particular, they are not true for strictly singular and strictly cosingular operators. See examples 1 and 2 in [8].

4. Theorems on particular operators. In beginning we define an AWC-cosingular operator. AWC is an abbreviation for almost weakly compact.

Definition 4.1. If $\mathscr{G}$ is the class of non-reflexive Banach spaces in definition 0.2, we call T an *AWC-cosingular operator.*

PROPOSITION 4.2. *If $T: X \to Y$ is weakly compact, then T is AWC-cosingular.*

Proof. If T is weakly compact, T' is weakly compact and hence T' is almost weakly compact. Now by proposition 3.1 (a), T is an AWC-cosingular operator.

COROLLARY 4.3. *Let $T: X \to Y$ and M any non-reflexive Banach space. If T is weakly compact, then T is M-cosingular.*

Remark 5. (a) The statement of corollary 4.3 is also true for M-singular operators (M non-reflexive), for if T is weakly compact, T is almost weakly compact and hence M-singular by remark 3.

(b) For uc operators, a characterization is given in proposition 1 of [8]. This can be stated as: $T: X \to Y$ is an l_1-cosingular operator if and only if T' is a uc operator. Another result on uc operators is the following

PROPOSITION 4.4. *Let $T: X \to Y$ and suppose Y is a separable Banach space. Then if T is a c_0-cosingular operator, T is also a uc operator.*

Proof. We show that if T is not a uc operator, then T is not a c_0-cosingular operator. So assume T is not a uc operator. Then there exists in X a subspace E isomorphic to c_0 and such that $T|E$ is an isomorphism between E and $T(E)$. Therefore, there exists an isomorphism, say i, between $T(E)$ and c_0. Since Y is separable there exists a continuous linear projection p from Y onto its subspace $T(E)$ isomorphic to c_0. This follows from theorem 4 of [9]. Let $h_2 = i \cdot p: Y \to c_0$ and $h_1 = h_2 \cdot T: X \to c_0$. Then

h_1 and h_2 are the required epimorphisms. Hence T is not a c_0-cosingular operator.

It is now shown that proposition 4.4 is not true in general for an arbitrary Y.

EXAMPLE 4.5. *If T is c_0-cosingular, then T is not necessarily a uc operator.*

Proof (see example 2 of [8]). Let $J: c_0 \to c_0'' = l_\infty$ be the canonical embedding. Then by proposition 5 of [7], the operator J is strictly cosingular, hence c_0-cosingular. But, clearly, J is not a uc operator since J has a bounded inverse on c_0.

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