COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

A NOTE ON SMOOTH FANS

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In his paper on fans, Charatonik [2] has characterized the fans embeddable in the Cantor fan as being folding. The question is raised on p. 28 whether smooth fans are folding. In a discussion with Dr. Charatonik, I was able to answer this question affirmatively by means of a recent theorem of Carruth [1]. In this note, I present the solution.

First some terms are defined. A *dendroid* is a hereditarily unicoherent arcwise connected metric continuum. A point t in a dendroid X is a *ramification point* of X provided at least three arcs in X with t as a common endpoint are pair-wise disjoint except for t. A *fan* is a dendroid with precisely one ramification point t. A fan X is *folding* provided there is a continuous function $f\colon X \to [0,1]$ such that for each x in X, the restriction of f to xt, the arc from x to t, is 1-1. A fan X is *smooth* provided $x_n \to x$ implies $x_n t \to xt$, where $x_n t \to xt$ means $\limsup x_n t = \liminf x_n t = xt$.

We shall make use of the weak cutpoint orderings of hereditarily unicoherent continua, first defined by Koch and Krule in [3]. Given a hereditarily unicoherent continuum X and a fixed t in X, the *weak cutpoint ordering of X relative to t*, $\leqslant_t$, is defined by $x \leqslant_t y$ if and only if $x \in yt$, where yt is the intersection of all subcontinua of X containing t and y. If X is a dendroid, it is readily seen that $\leqslant_t$ is a partial ordering of X. A dendroid X is called a *generalized tree* provided $\leqslant_t$ has a closed graph in $X \times X$ for some t. The following theorem is proved in [3], p. 680:

THEOREM 1. *A dendroid X is a generalized tree if and only if for some t in X, $x_n \to x$ implies $x_n t \to xt$.*

COROLLARY 2. *If X is a fan with ramification point t, then X is a generalized tree if and only if X is smooth.*

Proof. This immediately follows from Theorem 1 and the definition of a smooth fan.

Now we state the theorem of H. Carruth, which is proved in [1], p. 2:

THEOREM 3. *Let X be a compact metric space and let $\leqslant$ be a partial ordering of X with a closed graph in $X \times X$. Then there exists an order-*

preserving homeomorphism from X into the Hilbert cube Q, where Q is given the product ordering $(x_i) \leqslant (y_i)$ iff $x_i \leqslant y_i$ for $i = 1, 2, \ldots$

COROLLARY 4. *Smooth fans are folding.*

Proof. Let X be a smooth fan with ramification point t. By Corollary 2, $\leqslant_t$ has a closed graph. By Theorem 3, there is an order-preserving homeomorphism $h\colon X \to Q$. Define $f\colon X \to [0, 1]$ by $f(x) = d\big(h(x), h(t)\big)$, where d is the metric on Q given by

$$d\big((x_i), (y_i)\big) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

It is readily verified that f is continuous and that the restriction of f to xt is 1-1 for each x in X. Thus X is folding.

REFERENCES

[1] J. H. Carruth, *A note on partially ordered compacta*, Pacific Journal of Mathematics 24 (1968), p. 229-231.

[2] J. J. Charatonik, *On fans*, Dissertationes Mathematicae, Warszawa 1967.

[3] R. J. Koch and I. S. Krule, *Weak cutpoint ordering of hereditarily unicoherent continua*, Proceedings of the American Mathematical Society 11 (1960), p. 679-681.

Reçu par la Rédaction le 20. 12. 1967