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A MODEL WITH EXACTLY ONE UNDEFINABLE ELEMENT*

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THEOREM. *There is a relational structure $\mathfrak{A}$ which has finitely many fundamental relations and is such that exactly one of the elements of its universe is not first-order definable* (¹).

$\mathfrak{A}$ can be chosen so that its first-order theory is recursively axiomatizable. The question whether $\mathfrak{A}$ can even be chosen so that its theory is finitely axiomatizable remains open.

The structure $\mathfrak{A}$ will be of the form $\langle A, S, E\rangle$ where

(i) A is the (disjoint) union of the set N of finite ordinals and a set M of subsets of N;

(ii) S is the successor relation on N, i.e. $S = \{\langle n, n+1\rangle : n \in N\}$;

(iii) E is the membership relation between elements of N and the elements of M, i.e.

$$E = \{\langle n, s\rangle : n \in N \ \&\ s \in M \ \&\ n \in s\}.$$

Thus in order to specify $\mathfrak{A}$ we only have to choose M. Note that each element of N is definable in $\mathfrak{A}$; also each element of M which is periodic (²) (or differs from a periodic set by finitely many elements) is definable in $\mathfrak{A}$. We now prepare for the definition of M.

LEMMA 1. *Let $\langle\langle x_n, y_n\rangle : n \in N\rangle$ be an enumeration of all disjoint pairs of finite subsets of N. There is a sequence $\langle s_n : n \in N\rangle$ of subsets of N satisfying the following conditions*:

(1) *s_n is periodic with minimal period, say, p_n*;

(2) *p_n (properly) divides p_{n+1}*;

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(¹) The question whether such structures exist was raised in a seminar of J. Mycielski and C. Ryll-Nardzewski at the University of Wrocław. The author learnt about this problem through Jan Mycielski and is indebted to him for pointing out the flaws in earlier attempts to solve the problem.

(²) A subset A of N is *periodic* with period p for all $\nu \in N$, $\nu \in A$ if and only if $\nu + p \in A$.

(3) $x_n \cup y_n \subseteq p_{2n}(=\{0, 1, \ldots, p_{2n}-1\})$, $x_n \subset s_{2n}$, *and* $y_n \cap s_{2n} = \emptyset$;

(4) s_{2n+3} *begins with two copies of the initial period of* s_{2n+1}, i.e.

$$s_{2n+3} \cap \{0, \ldots, 2p_{2n+1}-1\}$$
$$= (s_{2n+1} \cap p_{2n+1}) \cup \{p_{2n+1}+\nu : \nu \in s_{2n+1}, \nu < 2p_{2n+1}\};$$

(5) *For each finite interval* x *of* N *of length* n *and each subset* z *of* x *there is an interval* y *of* N *with* $y \subseteq p_n$ *and for which the structures* $\langle x, \leqslant, s_0 \cap x, \ldots, s_{n-1} \cap x, z\rangle$ *and* $\langle y, \leqslant, s_0 \cap y, \ldots, s_n \cap y\rangle$ *are isomorphic.*

The proof proceeds via induction on n. In the choice of s_n one first takes care of (3) and (4), then of (5), and finally of (2) and (1).

Let $\langle s_n : n \in N\rangle$ be a sequence as obtained from Lemma 1. Put $s_\omega = \bigcup \{s_{2n+1} \cap p_{2n+1} : n \in N\}$, and define $M = \{s_0, \ldots, s_\omega\}$.

LEMMA 2. *For each finite subset* x *of* N *there is an* $n \in N$, *in fact infinitely many, for which* $x \subseteq p_n$, *and* $s_\omega \cap p_n = s_n \cap p_n$. *Hence, in view of Lemma* 1.2, s_ω *is not periodic, and thus distinct from* $s_0, s_1, \ldots$

We still have to prove that s_ω is not definable in $\mathfrak{A}$. For this purpose we prepare for the elimination of quantifiers in the first-order theory of (a definitorial expansion of) $\mathfrak{A}$.

LEMMA 3. *For any finite interval* x *of* N, *any finite sequence* $\langle t_0, \ldots, t_n\rangle$ *of elements of* M, *and any finite subset* z *of* N *there is an interval* y *of* N *with* $y \cap z = \emptyset$ *and being such that the structures* $\langle x, \leqslant, t_0 \cap x, \ldots, t_n \cap x\rangle$ *and* $\langle y, \leqslant, t_0 \cap y, \ldots, t_n \cap y\rangle$ *are isomorphic.*

The proof is based on Lemma 1.1 and, in case s_ω is involved, on 1.2, 1.4 and Lemma 2.

LEMMA 4. *Given a finite interval* x *of* N *of length* n *and a sequence* $\langle z_0, \ldots, z_m\rangle$ *of subsets of* x. *For any sequence* $\langle t_0, \ldots, t_m\rangle$ *of distinct elements of* M *other than* $s_0, \ldots, s_n$ *there is an interval* y *of* N *such that the structures* $\langle x, \leqslant, s_0 \cap x, \ldots, s_n \cap x, z_0, \ldots, z_m\rangle$ *and* $\langle y, \leqslant, s_0 \cap y, \ldots, s_n \cap y, t_0 \cap y, \ldots, t_m \cap y\rangle$ *are isomorphic.*

The proof is based on Lemma 1.5 and, in case s_ω is involved, also on Lemma 2.

LEMMA 5. *Given a pair* $\langle x, y\rangle$ *of disjoint finite subsets of* N *and a finite subset* X *of* M. *There is an* $n \in N$ *such that* $s_n \notin X$, $x \subseteq s_n$, $y \cap s_n = \emptyset$, *and* $x \cup y \subseteq p_n$.

The proof is based on Lemma 1.3.

Let $\mathfrak{A}^*$ be obtained from $\mathfrak{A}$ by expanding $\mathfrak{A}$ with the following definable relations and distinguished elements:

(i) each element of N and each element of M other than s_ω as a distinguished element;

(ii) N and M as one-place relations;

(iii) for each integer g the relation $S^g = \{\langle n, m\rangle : n \in N \ \& \ m \in N \ \& \ m = n+g\}$;

(iv) for each integer g the relation $S^g E = \{\langle n, s\rangle :$ for some m $\langle n, m\rangle \in S^g \ \& \ \langle m, s\rangle \in E\}$.

Clearly an element of A is definable in $\mathfrak{A}$ if and only if it is definable in $\mathfrak{A}^*$.

LEMMA 6. *In the first-order theory of $\mathfrak{A}^*$ quantifiers can be eliminated.*

By general reasoning on elimination of quantifiers it suffices to consider formulas of the form $\exists v\Phi$, where Φ is a conjunction of atomic or negated atomic formulas. Since N and M partion the universe we may assume that one of two cases holds: (a) Nv is a conjunct of Φ; (b) Mv is a conjunct of Φ. In case (a) we may further assume that Φ describes a situation as in Lemma 3. The interval x is an interval around v, the length of x being determined by the g's occurring in the atomic formulas of Φ. In addition the "variable elements" belonging to M may be required to be distinct from $s_0, \ldots, s_n$, n the length of x. By Lemma 3 we can ignore conjuncts of the form $\neg vS^g\ldots$ and by Lemma 4 we can settle the truth-value of the rest. In case (b) we can assume that Φ describes the situation of Lemma 5 and can settle the truth-value on the basis of that lemma.

That s_ω is indeed not definable in $\mathfrak{A}$ now follows from Lemma 6 and Lemma 2.

In a similar way one can show that given a finite cardinal $\mathfrak{m}$ there is a relational structure $\mathfrak{A}$ with finitely many relations and a $\mathfrak{m}$-element subset X of the universe of $\mathfrak{A}$ such that every $\mathfrak{m}$-element subset of the universe of $\mathfrak{A}$ is definable in $\mathfrak{A}$ if and only if it does not intersect X.

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