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ON CERTAIN CHARACTERIZATIONS OF DISTRIBUTIVE LATTICES

BY

K. GŁAZEK (WROCŁAW)

The purpose of this note is to give two characterizations of distributive lattices with zero and unity: first in the class of semirings and then in the class of positive semirings.

$(S; +, \cdot)$ is a *semiring* if two binary associative operations are defined on a set S, addition $+$ and multiplication $\cdot$, and if a two-sided law of distributivity of the multiplication with regard to the addition is satisfied:

$$a(b+c) = ab+ac, \qquad (b+c)a = ba+ca$$

for every $a, b, c \in S$.

The notion of a semiring was first introduced by Vandiver in [5]. He also gave examples of semirings which cannot be imbedded in a ring [6]. Obviously, every ring and every distributive lattice is a semiring.

The sets of natural numbers, of cardinal numbers not greater than a certain fixed cardinal number $\mathfrak{m}$, of endomorphisms of an arbitrary additive medial semigroup [2], of continuous real non-negative functions defined on a topological space, of non-negative upper (resp. lower) semicontinuous real functions defined on a topological space, each considered together with usual operations, are further examples of semirings.

The set of ideals of a commutative ring with complex operations (see [7], p. 22-23), and a meta-abelian group with commutation as the second operation are also semirings.

In Theorem 1 we shall assume that both semigroups $(S; +)$ and $(S; \cdot)$ have neutral elements zero and unity, i.e. elements 0 and $e \in S$ such that

$$0+a = a+0 = a = ea = ae \quad \text{for every } a \in S.$$

It is worthwhile to remark that neutral element with regard to addition in a semiring need not to have the property of zero with regard to multiplication. This can be seen on a set $S = \{0, a\}$ with operations

$+$	0	a
0	0	a
a	a	a

$\cdot$	0	a
0	a	a
a	a	a

Clearly, if a semiring has unity e, then element 0 must be the multiplicative idempotent.

THEOREM 1. *A semiring with* 0 *and* e *is a distributive lattice (with zero* 0 *and unity* e*) if and only if the following conditions are satisfied for every* $a, b, c, d \in S$:

(i) $e + a \cdot 0 = a \cdot 0 + e = e$,

(ii) $ab + cd = (a+c) \cdot (a+d) \cdot (b+cd)$.

Proof. Necessity of (ii) follows from the distributivity of addition with regard to multiplication.

In order to prove the sufficiency we need only to show, in view of a theorem of Birkhoff [1], p. 135, that for every $a \in S$ we have

(a) $$a^2 = a,$$

(b) $$e + a = a + e = e.$$

Putting $b = e$ and $c = d = 0$ in (ii) we obtain $a + 0^2 = a^2(e + 0^2)$. Therefore, for every $a \in S$, $a = a^2$. Hence (a) is proved.

Setting now $a = b = e$ and $d = 0$ in (ii) we get $e + c \cdot 0 = (e+c)(e \times + c \cdot 0)$. Hence, taking (i) into consideration, we have $e = e + c$ for every $c \in S$. If we put $b = 0$ and $c = d = e$ in (ii), we obtain $a \cdot 0 + e = (a+e) \times \times (a+e) \cdot (a \cdot 0 + e)$, whence, making use of (i) and of the already proved condition (a), we get $e = a + e$. Therefore condition (b) is also proved. Thus the proof of Theorem 1 is completed.

Note that condition (i) in Theorem 1 does not follow from other assumptions. A simple example to show this is the set $S = \{0, a, e\}$ with the following operations:

$+$	0	a	e
0	0	a	e
a	a	a	a
e	e	a	e

$\cdot$	0	a	e
0	0	a	0
a	a	a	a
e	0	a	e

THEOREM 2. *A semiring with e is a distributive lattice (with unity e) if and only if the following conditions are satisfied for each $a \in S$:*

(i) $e+a = a+e$;

(ii) *there exists an inverse element to $e+a$;*

(iii) *there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$.*

Proof. The necessity is obvious. To prove the sufficiency, we have, as in the proof of Theorem 1, only to show (a) and (b).

First let us prove that $2a = a$ for each $a \in S$. In fact, if $a^n = a$ for some $n > 1$, then

$$(a^{n-1})^2 = a^{n+n-2} = aa^{n-2} = a^{n-1}.$$

Therefore a^{n-1} is idempotent. Hence and from (iii) we infer that $(e+e)^m$ is idempotent for some $m \geqslant 1$. But $(e+e)^m = (2^m-1)e+e$, whence, by applying (i), we get $(e+e)^m = e$, because, as it is easy to see, if $e+a$ is idempotent, then $e+a = e$ by (ii). Now, if $l > 1$ is the last integer such that $le = e$, then

$$[(l-1)e]^2 = (l^2-2l)e+e = (l-2)e+e = (l-1)e.$$

Hence, $(l-1)e = e$. We infer that $l = 2$ and that $a+a = a(e+e) = ae = a$, whence $2a = a$.

Taking this into account we have

$$(e+a)^n = e+a+\ldots+a^{n-1}+a = e+a+\ldots+a^{n-1} = (e+a)^{n-1},$$

because addition of powers of any element a is, by (i), commutative. Hence, by (ii), $e+a = e$, that is, condition (b) is proved.

Now, if $k > 1$ is the least integer such that $a^k = a$, then

$$a^{k-1} = ea^{k-1} = (a+e)a^{k-1} = a+a^{k-1} = a(e+a^{k-2}) = ae = a.$$

Hence $k = 2$ and in this way we come to (a), which completes the proof of Theorem 2.

It may be of interest to notice that condition (iii) of Theorem 2 for rings implies commutativity (see [3], p. 217).

Słowikowski and Zawadowski [4] have defined and investigated the so-called *positive semirings*. A semiring with unity and zero is positive if condition (ii) of Theorem 2 holds for each a and if both operations are commutative. This leads to a simple

COROLLARY. *A positive semiring S is a distributive lattice (with zero and unity) if and only if for each $a \in S$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$.*

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

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