COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 2

# STRUCTURE AND ORDER STRUCTURE IN ABELIAN GROUPS

BY

ANNE C. MOREL (SEATTLE, WASH.)

**Introduction.** As is well known, an Abelian group can be linearly ordered if and only if it is torsion-free. The main point of this note (1) is to give criteria for determining what types of order a given Abelian group admits. It turns out that the types of order depend essentially on the least upper bound of the number of generators of those free Abelian groups that are pure subgroups.

Notation and definitions on order types will be taken from [6]; definitions relevant to ordered groups can be found in [3]. Let $\varphi$ be an ordinal. We denote by $(\omega^*+1+\omega)_0^\varphi$ the order type of the relation $R$ whose field is the set of all $\varphi$-termed sequences of integers that are almost always 0; for any two such sequences $a$ and $b$, $aRb$ if and only if $a$ precedes $b$ in antilexicographic order. Note that $(\omega^*+1+\omega)_0^\varphi = (\omega^*+\omega)^\varphi$ for finite ordinals $\varphi$. For any set $A$, $\varkappa(A)$ denotes the cardinality of $A$. Moreover, let $\mu$ be an order type, let $R$ be any relation with $\tau(R) = \mu$; it is convenient to use the symbol $\varkappa(\mu)$ to denote the cardinality of $F(R)$. In particular, for every ordinal $\theta$, $\varkappa(\theta)$ is thus defined. The symbol $+$ will be used in three senses: ordinal addition, cardinal addition, and the group operation. The symbol $\cdot$ will be used for ordinal multiplication and for cardinal multiplication. It is believed that context will always allow us to distinguish. All groups considered are Abelian; hence the word *Abelian* will be omitted. By a *proper subgroup* of $\mathfrak{G} = (G, +)$ we mean a subgroup $\mathfrak{H} = (H, +)$ where $\emptyset \subset H \subset G$. An *ordered group* is an ordered triple $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$, where

(i) $(G, +)$ is a group,

(ii) $\leqslant$ is a linear ordering relation with $G = F(R)$,

(iii) for all $a, b, c \in G$, $a \leqslant b$ implies $a+c \leqslant b+c$.

If $(H, +)$ is a subgroup of $(G, +)$ and if $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ is an ordered group, then $\mathfrak{O}\mathfrak{H} = (H, +, \leqslant_H)$, where $\leqslant_H = \leqslant \langle H\rangle$, is an

(1) Written with support from N.S.F. Grant GP-5691. The author is indebted to R. S. Pierce for several helpful suggestions.

*ordered subgroup* of $\mathfrak{O}\mathfrak{G}$. If $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ is an ordered group, and if $\tau(\leqslant\langle G\rangle) = a$, we shall say that $\mathfrak{G} = (G, +)$ *admits an ordering of type* $a$. If $\mathfrak{O}\mathfrak{H} = (H, +, \leqslant)$ is an ordered subgroup of $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ and if whenever $a < x < b$ and $a, b \in H$, then $x \in H$, then $\mathfrak{O}\mathfrak{H}$ is a *convex ordered subgroup*. Note that if $\mathfrak{O}\mathfrak{H}$ is a convex ordered subgroup of $\mathfrak{O}\mathfrak{G}$, then $\mathfrak{H}$ is a pure subgroup of $\mathfrak{G}$. For every cardinal $a$, $\mathfrak{F}_a$ denotes the free group with $a$ generators. It is convenient to assume that $\mathfrak{F}_0$ is a one-element group; hence $\mathfrak{F}_0$ admits the order type 1. A group not isomorphic to $\mathfrak{F}_0$ will be called *non-trivial*. Isomorphism and identity will be freely confused. The additive group of rational numbers, with the usual ordering, will be written as $\mathfrak{O}\mathfrak{R} = (R, +, \leqslant)$, with $\eta$ for $\tau(\leqslant)$. For any group $\mathfrak{G} = (G, +)$, we write $\varkappa(\mathfrak{G})$ for $\varkappa(G)$; hence if $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$, the cardinality of the quotient group is $\varkappa(\mathfrak{G}/\mathfrak{H})$. The symbol $r(\mathfrak{G})$ will be used for the rank of the group $\mathfrak{G}$.

LEMMA 1. *Let $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ be an ordered group with $\tau(\leqslant) = a$. If $a$ is a scattered type, then*

$$a = (\omega^* + 1 + \omega)_0^{\varphi}$$

*for a unique ordinal $\varphi$.*

Proof. See [7], p. 214, Corollary 1, and p. 213, Theorem.

The simple lemma below will be used several times.

LEMMA 2. *Let $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ be an ordered group, let $a, b, c \in G$. Then $\leqslant\langle[a, b]\rangle \cong \leqslant\langle[a+c, b+c]\rangle$.*

THEOREM 3. *Let $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ be an ordered group, let $a = \tau(\leqslant)$. Then $a$ satisfies exactly one of the following conditions:*

(i) *$a = (\omega^* + 1 + \omega)_0^{\varphi}$ for a unique ordinal $\varphi$.*

(ii) *$a = (\omega^* + 1 + \omega)_0^{\varphi} \cdot \delta$ for a unique non-zero ordinal $\varphi$ and some dense order type $\delta$.*

(iii) *$a$ is a dense type* (2).

Proof. If $a$ is scattered, then $a$ satisfies (i) by Lemma 1. Now assume that $a$ is not scattered. First, consider the following case:

(1) For every $a \in G$ such that $0 < a$ there exists a $c \in G$ such that $0 < c < a$.

(2) The theorem was obtained for denumerable (not necessarily Abelian) ordered groups by Mal'cev (see [5] and [4]); it is evident that Theorem 3 applies to non-Abelian groups also. A closely related result is [12], p. 18, Theorem 4.

A classification of the dense types occurring in (i) (or, equivalently, in (iii)) appears to be a difficult problem; the problem seems to us to be too specialized to be worthwhile.

A coarser classification of types of order is between *discrete* and *dense*, the discrete types being those of (i) and (ii). The latter classification appears to be appropriate for the study of the first-order theory of ordered groups (see [2]), while ours is suitable for the investigation of the algebraic properties.

It is apparent that $\leqslant$ is a dense relation, for assume that $d, e \in G$ and $d < e$; then $0 < e < d$; hence there is an $f$ such that $0 < f < e-d$, and so $d < f+d < e$.

Now, suppose that (1) does not hold. Then there exists a "smallest" element $e$ such that $0 < e$. By [6], p. 53, Corollary 3.3,

(2) $\leqslant = \sum_{i,R} S_i$, where $R$ is a dense relation and each $S_i$ is a scattered subrelation of $\leqslant$.

Let $S_0$ be the unique $S_i$ for which $0 \in F(S_i)$; let $H = F(S_0)$. Clearly

$$e \in H. \tag{3}$$

If $a, b \in H$, then, using (2), we see that

(4) $\leqslant \langle [a, b] \rangle$ is a scattered relation.

By Lemma 2,

$$\leqslant \langle [a, b] \rangle \cong \leqslant \langle [0, b-a] \rangle;$$

hence, by (2) again, $b-a \in H$. Employing (3), one now obtains the conclusion that $\mathfrak{O}\mathfrak{H} = (H, +, \leqslant_H)$ is an infinite convex subgroup of $(G, +, \leqslant)$. With the help of (2), we conclude that the interval $S_0$ is scattered. Now by Lemma 1,

$$\tau(S_0) = (\omega^* + 1 + \omega)_0^\varphi \quad \text{for some } \varphi > 0.$$

Now suppose $S_i \neq S_0$. Pick an element $a \in F(S_i)$. It is easy to see that the function

$$f(x) = x + a \quad \text{for all } x \in F(S_0)$$

maps the relation $S_0$ isomorphically onto $S_i$; for, if this were not the case, a scattered relation would be isomorphic to a non-scattered relation. Now with the help of (2) one obtains that

$$\tau(\leqslant) = (\omega^* + 1 + \omega)_0^\varphi \cdot \delta \quad \text{with} \quad \delta = \tau(R).$$

The cases (i), (ii), and (iii) are clearly mutually exclusive.

From the proof of Theorem 3, we obtain

COROLLARY 4. *Let* $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$, *where* $\tau(\leqslant) = (\omega^* + 1 + \omega)_0^\varphi \cdot \delta$, *and* $\delta$ *is a dense type. Then* $\mathfrak{O}\mathfrak{G}$ *has a convex ordered subgroup,* $\mathfrak{O}\mathfrak{H} = (H, +, \leqslant_H)$ *with* $\tau(\leqslant_H) = (\omega^* + 1 + \omega)_0^\varphi$; *moreover,* $\mathfrak{G}/\mathfrak{H}$ *admits the ordering* $\delta$.

LEMMA 5. *Let* $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ *be an ordered group with* $\tau(\leqslant) = (\omega^* + 1 + \omega)_0^\varphi$. *For each ordinal* $0 < \varphi$ *there exists a convex subgroup*

$$\mathfrak{O}\mathfrak{H}_\theta = (H_\theta, +, \leqslant_{H_\theta})$$

*such that*

(i) $\tau(\leqslant\langle H_\theta\rangle) = (\omega^*+1+\omega)_0^\theta$,

(ii) $\mathfrak{G}/\mathfrak{H}_\theta$ *admits an ordering of type* $(\omega^*+1+\omega)_0^{\chi_\theta}$, *where* $\chi_\theta$ *is the unique ordinal satisfying the equation* $\theta+\chi_\theta = \varphi$.

Proof. By hypothesis, the elements of $G$ can be labelled as

$$\{a \mid a \text{ is on } \varphi \text{ to the integers and } a_\iota = 0 \text{ for almost all } \iota < \varphi\}.$$

Then $a \leqslant b$ if and only if $a$ precedes $b$ in antilexicographic ordering.

Since (see [7], p. 214, Corollary 2) the relation $\leqslant$ is one-point homogeneous, we can designate the zero element of $G$ as that sequence $a$ for which $a_\iota = 0$ for all $\iota < \varphi$.

We need the following statement:

(1) Suppose $b, b' \epsilon G$ and $b < b'$. Let $\mu$ be the largest ordinal for which $b_\mu \neq b'_\mu$. Then the order type of $\leqslant\langle(b, b')\rangle$ is

$$\sum_{\iota<\mu}\big((\omega^*+1+\omega)_0^\iota\cdot\omega\big)+(\omega^*+1+\omega)_0^\mu\cdot\big((b'_\mu-b_\mu)-1\big)+\Big(\sum_{\iota<\mu}\big((\omega^*+1+\omega)_0^\iota\cdot\omega\big)\Big)^*.$$

For brevity, we write the type above as $(\mu; n)$, where $n = (b'_\mu-b_\mu)-1$.

The proof of (1) will not be given here; it easily follows from [12], p. 16, Theorem 4. We also make use of the following fact:

(2) If $(\mu; n) = (\mu'; n')$, then $\mu = \mu'$ and $n = n'$.

The proof of (2) will also be omitted; it is not too short, and the technique is not relevant to the material of this note.

Now let

$$H_\theta = \{a \mid a \epsilon G \text{ and } a_\iota = 0 \text{ for all } \iota \text{ with } \theta \leqslant \iota < \varphi\}. \tag{3}$$

Obviously

$$\tau(\leqslant\langle H_\theta\rangle) = (\omega^*+1+\omega)_0^\theta.$$

Now let $b, c \epsilon H_\theta$; for definiteness, assume that $0 < b$.

Let $d = b+c$; we want to show that

$$d_\iota = 0 \quad \text{for all } \iota \geqslant \theta. \tag{4}$$

Suppose that (4) is false; suppose $\mu \geqslant \theta$ is the largest ordinal $\iota$ for which $d_\iota \neq 0$. Now, by using Lemma 2,

$$\leqslant\langle(b, b+c)\rangle \cong \leqslant\langle(0, c)\rangle.$$

Hence

$$\begin{cases} \tau\big(\leqslant\langle(b, b+c)\rangle\big) = (\mu; m) \text{ for some } m; \\ \tau\big(\leqslant\langle(0, c)\rangle\big) = (\varrho; n) \text{ for some } \varrho < \theta \text{ and for some } n. \end{cases} \tag{5}$$

Using (5) and (2) one obtains a contradiction. Thus (4) holds.

Now suppose $b \in H_\theta$; assume $0 < b$. From Lemma 2,

$$\leqslant \langle(-b, 0)\rangle \cong \leqslant \langle(0, b)\rangle;$$

with the help of (1), (2), and (3), we see that $-b \in H_\theta$. Hence we get (i).

To prove (ii), note that, for any $a, b \in G$, $a = b \pmod{\mathfrak{H}_\theta}$ if and only if $a_\iota = b_\iota$ for all $\iota \geqslant \theta$. Moreover, it is easy to see that $a/\mathfrak{H}_\theta$ is an interval of $\leqslant$. Now for every coset $a/\mathfrak{H}_\theta$ put

$$f(a/\mathfrak{H}_\theta) = \{(\iota, a_{\theta+\iota}) \mid 0 \leqslant \iota < \chi_\theta\};$$

and put

$a/\mathfrak{H}_\theta \leqslant' b/\mathfrak{H}_\theta$ if and only if $a/\mathfrak{H}_\theta = b/\mathfrak{H}_\theta$ or $f(a/\mathfrak{H}_\theta)$ precedes $f(b/\mathfrak{H}_\theta)$ in antilexicographic ordering.

It is easy to check that $\mathfrak{G}/\mathfrak{H}_\theta$ admits ordering $\tau(\leqslant')$ and that (ii) holds.

Lemma 6. *Let $\mathfrak{G}$ be any group. Then the following conditions are equivalent*:

(i) $\mathfrak{G} \cong \mathfrak{F}_1$.

(ii) *The only type of ordering that $\mathfrak{G}$ admits is $\omega^* + \omega$.*

(iii) *$\mathfrak{G}$ admits an ordering of type $\omega^* + \omega$.*

Proof. All the implications are obvious except for the statement that (i) implies (ii). Assume that $\mathfrak{F}_1$ admits the following ordering:

(1) $$\mathfrak{O}\mathfrak{F}_1 = (F_1, +, \leqslant),$$

where $\tau(\leqslant) = (\omega^* + \omega) \cdot \beta$ for some $\beta \neq 1$.

From Corollary 4 and Lemma 5, it follows that $\mathfrak{O}\mathfrak{F}_1$ has a proper convex subgroup, hence a proper pure subgroup; but $\mathfrak{F}_1$ has no proper pure subgroup.

Now assume that

(2) $$\mathfrak{O}\mathfrak{F}_1 = (F, +, \leqslant), \qquad \text{where} \qquad \tau(\leqslant) = \eta.$$

Assume that $\mathfrak{F}_1$ is the set of integers and that $0 < 1$ in the ordering $\leqslant$. By (2), there must exist an $a \in \mathfrak{F}_1$ such that

$$0 < a < 1.$$

However, $\mathfrak{F}_1$ is generated by 1, thus no positive element of $\mathfrak{F}_1$ can be less than 1. Hence (2) is also impossible, and we conclude (ii).

LEMMA 7. *Let $\mathfrak{G}$ be a group, let $a$ be a cardinal. Then the following conditions are equivalent*:

(i) $\mathfrak{G} \cong \mathfrak{F}_a$.

(ii) $\mathfrak{G}$ *admits all orderings of type* $(\omega^* + 1 + \omega)_0^\varphi$, *where* $\varkappa(\varphi) = a$ (3).

Proof. Suppose (i) holds and $\varkappa(\varphi) = a$. Then we can label a set of free generators of $G$ as

$$\{a_\iota \mid 0 \leqslant \iota < \varphi\}.$$

Now using a natural antilexicographic ordering of the elements of $G$, we obtain (ii).

In order to prove that (ii) implies (i), we note that a weaker statement is proved in [11], namely

(1) If $\mathfrak{G}$ admits an ordering of type $(\omega^* + 1 + \omega)_0^\varphi$ for some $\varphi$, then $\mathfrak{G} \cong \mathfrak{F}_a$ for some $a$.

We note that

(2) If $0 < \varkappa(\varphi) \leqslant \aleph_0$, then $\varkappa\big((\omega^* + 1 + \omega)_0^\varphi\big) = \aleph_0$,

and also

(3) If $\aleph_0 < \varkappa(\varphi)$, then $\varkappa\big((\omega^* + 1 + \omega)_0^\varphi\big) = \varkappa(\varphi)$.

If $\aleph_0 < \varkappa(\varphi)$, then (i) follows from (3) and (1). In order to obtain (i) when $\varkappa(\varphi) \leqslant \aleph_0$, it is sufficient to prove two statements:

(4) If $\mathfrak{G}$ admits an ordering of type $(\omega^* + \omega)^n$, where $1 \leqslant n < \omega$, then $\mathfrak{G} \cong \mathfrak{F}_n$.

(5) If $\mathfrak{G}$ admits an ordering of type $(\omega^* + 1 + \omega)_0^\varphi$, where $\varkappa(\varphi) = \aleph_0$, then $\mathfrak{G} \cong \mathfrak{F}_{\aleph_0}$.

Now (4) holds for $n = 1$ by Lemma 6, (i), (iii). Assume that (4) holds for $n$ and that $G$ admits the ordering $(\omega^* + \omega)^{n+1}$. Hence we have

$$\mathfrak{O}\mathfrak{G} = (G, +, \leqslant), \quad \text{where} \quad \tau(\leqslant) = (\omega^* + \omega)^{n+1}.$$

By Lemma 5 there exists a convex, and hence pure, subgroup $\mathfrak{O}\mathfrak{H}$ for which

$$\mathfrak{O}\mathfrak{H} = (H, +, \leqslant_H) \quad \text{and} \quad \tau(\leqslant_H) = (\omega^* + \omega)^n,$$

and for which

(6) $\mathfrak{G}/\mathfrak{H}$ admits the ordering $\omega^* + \omega$.

---

(3) If one omits the tacit assumption that $\mathfrak{G}$ is Abelian, the theorem becomes false, i.e., for every cardinal $a > 2$, there exists a non-Abelian group admitting the orderings $(\omega^* + 1 + \omega)_0^\varphi$, where $\varkappa(\varphi) = a$; see [8].

By the inductive hypothesis, $\mathfrak{H} \cong \mathfrak{F}_n$; thus

$$r(\mathfrak{H}) = n.$$

Using (6) and Lemma 6 (i) and (iii), one gets

$$r(\mathfrak{G}/\mathfrak{H}) = 1.$$

Now using the familiar identity,

(7) $r(\mathfrak{G}) = r(\mathfrak{H}) + r(\mathfrak{G}/\mathfrak{H})$, when $\mathfrak{H}$ is a pure subgroup of $\mathfrak{G}$,

we see that

$$r(\mathfrak{G}) = n+1. \tag{8}$$

By (8) and (1), we have $G \cong \mathfrak{F}_{n+1}$, and so (4) holds.

Now suppose that $\mathfrak{G}$ satisfies the hypothesis of (5). By (1) and (2), either

$$\mathfrak{G} \cong \mathfrak{F}_{\aleph_0},$$

or

$$\mathfrak{G} \cong \mathfrak{F}_n \text{ for some finite } n. \tag{9}$$

Suppose (9) holds. Using (4) and Lemma 5, we find that $\mathfrak{G}$ has a pure subgroup $\mathfrak{H}$ with

$$\mathfrak{H} \cong \mathfrak{F}_{n+1},$$

in contradiction to (9). Thus (5) holds.

LEMMA 8. *Let $\mathfrak{G} = (G, +)$ be a torsion-free group and let $\mathfrak{H}$ be a pure subgroup of $\mathfrak{G}$. If $\mathfrak{H}$ admits the ordering $\beta$ and if $\mathfrak{G}/\mathfrak{H}$ admits the ordering $\gamma$, then $\mathfrak{G}$ admits the ordering $\beta\cdot\gamma$.*

Proof. By hypothesis

$$\mathfrak{O}\mathfrak{H} = (H, +, \leqslant_1), \quad \text{where} \quad \tau(\leqslant_1) = \beta,$$
$$\mathfrak{G}/\mathfrak{H} = (K, +, \leqslant_2), \quad \text{where} \quad \tau(\leqslant_2) = \gamma.$$

Now define a relation $\leqslant$ with $F(\leqslant) = G$ as follows: $a \leqslant b$ if $a/\mathfrak{H} <_2 b/\mathfrak{H}$ or if $a/\mathfrak{H} = b/\mathfrak{H}$ and $0 \leqslant_1 b-a$.

It is evident that $\mathfrak{O}\mathfrak{G} = (G, +, \leqslant)$ is an ordered group with $\tau(\leqslant) = \beta\cdot\gamma$.

THEOREM 9. *Let $\mathfrak{G}$ be a non-trivial torsion-free group. The following conditions are equivalent:*

(i) $\mathfrak{G} \neq \mathfrak{F}_1$.

(ii) $\mathfrak{G}$ *admits a dense ordering* [4].

---

[4] There exist non-Abelian ordered groups that do not admit a dense ordering; see [8].

Proof. By (i) and (ii) of Lemma 6, we obtain that implication (ii) implies (i). Now suppose $\mathfrak{G}$ non $\cong \mathfrak{F}_1$. It is easily seen that $\mathfrak{G}$ has a pure (not necessarily proper) subgroup $\mathfrak{H}$ with

(1) $$\aleph_0 \leqslant \varkappa(\mathfrak{H}) \leqslant 2^{\aleph_0} \quad \text{and} \quad \mathfrak{H} \text{ non} \cong \mathfrak{F}_1.$$

By [1], p. 226, Ex. 2(b), every torsion-free Abelian group of cardinality not greater than that of the continuum admits an Archimedean order. By [11], p. 24, Theorem 1, if $\mathfrak{O}\mathfrak{H} = (H, +, \leqslant)$, where $\tau(\leqslant)$ is a discrete type (see footnote (2)), and $\leqslant$ is an Archimedean ordering, then $\mathfrak{H} \cong \mathfrak{F}_1$, in contradiction to (1). Now using Theorem 3, we find that $\mathfrak{H}$ admits a dense ordering, say $\delta$. Since $\mathfrak{H}$ is a pure subgroup of $\mathfrak{G}$, it follows that $\mathfrak{G}/\mathfrak{H}$ must be torsion-free. From the familiar fact that every torsion-free group can be ordered (see, for example, [3], p. 36, Corollary 5), one obtains an ordering of some type $\beta$ for $\mathfrak{G}/\mathfrak{H}$. From Lemma 8, $\mathfrak{G}$ admits an order of type $\delta\cdot\beta$. Since $\delta$ is the type of a dense relation without first or last element, it is clear that $\delta\cdot\beta$ must be dense also.

THEOREM 10. *$\mathfrak{F}_a$ admits precisely the following types of order:*

(i) *All types $(\omega^*+1+\omega)_0^{\varphi}$, where $\varkappa(\varphi) = a$.*

(ii) *If $a$ is finite, all types of the form $(\omega^*+\omega)^m\cdot\eta$, where $1 \leqslant m \leqslant a-2$.*

(iii) *If $a$ is infinite, for each ordinal $\varphi$ with*

$$1 \leqslant \varkappa(\varphi) \leqslant a,$$

*and for each infinite cardinal $\varrho$ satisfying*

$$\max\big(\varkappa(\varphi), \varrho\big) = a,$$

*a type of the form*

$$(\omega+1+\omega)_0^{\varphi}\cdot\delta,$$

*where $\delta$ is dense and $\varkappa(\delta) = \varrho$.*

(iv) *Dense (for $a \geqslant 2$).*

Proof. By Lemma 7, $\mathfrak{F}_a$ admits the orderings of (i).

Now suppose that $a$ is finite and $1 \leqslant m \leqslant a-2$. Let $\mathfrak{H}$ be a pure subgroup of $\mathfrak{F}_a$, with $\mathfrak{H} \cong \mathfrak{F}_m$.

Referring to the familiar formula

(1) $$r(\mathfrak{G}) = r(\mathfrak{H}) + r(\mathfrak{G}/\mathfrak{H}) \quad \text{for every pure subgroup } \mathfrak{H},$$

we see that $r(\mathfrak{G}/\mathfrak{H}) \geqslant 2$, and hence

$$\mathfrak{G}/\mathfrak{H} \text{ non} \cong \mathfrak{F}_1.$$

Using Theorem 9 and Lemma 8 we obtain the orderings of (ii).

Now let $a, \varphi$ and $\varrho$ satisfy the premises of (iii). Let

$$\{a_\iota | \iota < \varkappa(\varphi) + \varrho\}$$

be a set of free generators of $\mathfrak{F}_\alpha$. Let $\mathfrak{H}$ be the (pure) subgroup generated by $\{a_\iota \mid \iota < \varkappa(\varphi)\}$.

Then $\mathfrak{H} \simeq \mathfrak{F}_{\varkappa(\varphi)}$; by Lemma 7, $\mathfrak{H}$ admits order $(\omega^* + 1 + \omega)_0^\varphi$. Clearly,

$$\mathfrak{F}_\alpha / \mathfrak{H} \simeq \mathfrak{F}_\varrho .$$

Since $\varrho$ is infinite, using Theorem 9 one obtains that $\mathfrak{F}_\alpha/\mathfrak{H}$ admits a dense ordering of type $\delta$, where $\varkappa(\delta) = \varrho$. Now using Lemma 8 one obtains the orderings of (iii). Obviously, (iv) follows from Theorem 9.

It remains to be proved that the types of order listed are the only possibilities. That there are no other scattered orderings follows from Lemma 7 and from (2) and (3) of the proof of Lemma 7. Suppose that $\alpha$ is finite and that $\mathfrak{F}_\alpha$ admits an ordering of type.

$$(\omega^* + 1 + \omega)_0^\theta \cdot \eta, \qquad \text{where} \qquad 1 \leqslant \alpha - 1 \leqslant \theta .$$

Clearly, $\theta$ cannot be greater than $\alpha - 1$, for otherwise, by Corollary 4 and Lemma 5, $\mathfrak{F}_\alpha$ would have a pure proper subgroup isomorphic to $\mathfrak{F}_\theta$ where $\theta \geqslant \alpha$. If one puts $\theta = \alpha - 1$, then one easily obtains a contradiction with the help of Corollary 4, Theorem 9, the fact that $\mathfrak{F}_\alpha/\mathfrak{H}$ is free Abelian for every pure subgroup $\mathfrak{H}$, and (1). If $\alpha$ is infinite, then an obvious cardinality argument, using (2) and (3) of the proof of Theorem 7, precludes any other discrete types. Reference to Theorem 3 completes the proof.

THEOREM 11. *Suppose that*

(a) $\mathfrak{G}$ *is torsion-free but* $\mathfrak{G}$ *is not free;*

(b) $\alpha$ *is the smallest cardinal with the property that* $\mathfrak{G}$ *has no pure free subgroup isomorphic to* $\mathfrak{F}_\alpha$;

(c) $\varrho$ *is the smallest cardinal such that, for some pure free subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$, $\varkappa(\mathfrak{G}/\mathfrak{K}) = \varrho$.

*Then* $\mathfrak{G}$ *admits the following types of order:*

(i) *dense;*

(ii) *if* $\alpha$ *is finite, all types of the form* $(\omega^* + \omega)^n \cdot \delta$, *where* $n < \alpha$, $\delta$ *is dense and* $\varkappa(\delta) = \varkappa(\mathfrak{G})$;

(iii) *if* $\alpha$ *is infinite, all types of the form* $(\omega^* + 1 + \omega)_0^\varphi \cdot \delta$, *where*

(I) $0 < \varkappa(\varphi) < \alpha$,

(II) $\delta$ *is dense and* $\varrho \leqslant \varkappa(\delta)$,

(III) $\max\bigl(\varkappa(\varphi), \varkappa(\delta)\bigr) = \varkappa(\mathfrak{G})$.

Proof. Obviously, $\mathfrak{G}$ admits a dense ordering. To show that $\mathfrak{G}$ admits the orderings of (ii) and (iii), we need the following fact:

(1) If $\mathfrak{K}$ is a pure free subgroup of $\mathfrak{G}$, then $\mathfrak{G}/\mathfrak{K}$ is not free.

If (1) were false, it would follow from [2], p. 38, Theorem 9.2, that $\mathfrak{G}$ is a free group, in contradiction to (a).

If $a$ is finite, then, using (1), Theorem 9, and Lemma 8, we see that $\mathfrak{G}$ admits the orderings of type (ii).

Suppose that $a$ is infinite; in this case we make the convention that

$$\varphi \text{ is an ordinal and } \varphi < a.$$

First, consider the subcase

(2) $$\varkappa(\varphi) < \varkappa(\mathfrak{G}) \quad \text{or} \quad \varrho = \varkappa(\mathfrak{G}).$$

By hypothesis, there exists a pure subgroup $\mathfrak{K}$ of $\mathfrak{G}$ with

(3) $$\mathfrak{K} \cong \mathfrak{F}_{\varkappa(\varphi)};$$

hence, using (2), we get

(4) $$\varkappa(\mathfrak{G}/\mathfrak{K}) = \varkappa(\mathfrak{G}).$$

By (3), Lemma 7, Theorem 9, Lemma 8, and (4), it follows that $\mathfrak{G}$ admits an ordering of type

(5) $$(\omega^* + 1 + \omega)_0^{\varphi} \cdot \delta, \quad \text{where} \quad \varkappa(\delta) = \varkappa(\mathfrak{G}).$$

Now consider the subcase

(6) $$\varkappa(\varphi) = \varkappa(\mathfrak{G}) \quad \text{and} \quad \varrho < \varkappa(\mathfrak{G}).$$

By hypothesis, there exists a pure free subgroup $\mathfrak{K}$ of $\mathfrak{G}$ such that

(7) $$\varkappa(\mathfrak{G}/\mathfrak{K}) = \varrho,$$

and hence

$$\mathfrak{K} \cong \mathfrak{F}_{\varkappa(\mathfrak{G})}.$$

Let $\mu$ be any cardinal satisfying the inequality

(8) $$\varrho \leqslant \mu \leqslant \varkappa(\mathfrak{G}).$$

Since, by (6) and (8), $\varphi + \mu = \varphi$, we can label a set of free generators of $\mathfrak{K}$ as

$$\{a_\iota \mid \iota < \varphi + \mu\}.$$

Let $\mathfrak{L}$ be the subgroup of $\mathfrak{K}$ generated by $L = \{a_\iota \mid \iota < \varphi\}$. Obviously,

(9) $$\mathfrak{L} \text{ is a pure subgroup of } \mathfrak{G};$$

moreover,

(10) $$\mathfrak{L} \cong \mathfrak{F}_{\varkappa(\mathfrak{G})} \quad \text{and} \quad \varkappa(\mathfrak{K}/\mathfrak{L}) = \mu.$$

Since

$$\varkappa(\mathfrak{G}/\mathfrak{L}) = \varkappa(\mathfrak{G}/\mathfrak{K})\cdot\varkappa(\mathfrak{K}/\mathfrak{L}),$$

using (7), (8), and (10), we get

$$\varkappa(\mathfrak{G}/\mathfrak{L}) = \mu. \tag{11}$$

Using (9), (10), (11), Lemma 7, Theorem 9, and Lemma 8, we find that $\mathfrak{G}$ admits an ordering of type

$$(\omega^*+1+\omega)_0^{\varphi}\cdot\delta, \quad \text{where } \delta \text{ is dense and } \varkappa(\delta) = \mu. \tag{12}$$

By (5) and (12), all the types of (iii) have been obtained. Finally, a familiar argument using Corollary 4, Lemma 7, Lemma 5, (2) and (3) of Lemma 7, and Theorem 3 yields that all admissable types have been enumerated in (i), (ii), and (iii).

## *REFERENCES*


[1] G. Birkhoff, *Lattice theory*, Revised edition, New York 1948.

[2] L. Fuchs, *Abelian groups*, Budapest 1948.

[3] — *Partially ordered algebraic systems*, Oxford 1963.

[4] I. Kaplansky, *Review of* [5], Mathematical Reviews 11 (1950), p. 373.

[5] А. И. Мальцев, *Об упорядоченных группах*, Известия Академии Наук СССР, серия математическая, 13 (1949), p. 473-482.

[6] A. C. Morel, *On the arithmetic of order types*, Transactions of the American Mathematical Society 92 (1959), p. 48-71.

[7] — *A class of relation types isomorphic to the ordinals*, Michigan Mathematical Journal 12 (1965), p. 203-215.

[8] — *Remarks on absolutely convex subgroups*, Preliminary report, Notices of the American Mathematical Society 13 (1966), p. 100.

[9] A. Robinson and E. Zakon, *Elementary properties of ordered Abelian groups*, Transactions of the American Mathematical Society 96 (1960), p. 222-236.

[10] N. Sankaran, *Classification of totally ordered abelian groups*, Journal of the Indian Mathematical Society 29 (1965), p. 9-29.

[11] — and R. Venkataraman, *A generalization of the ordered group of integers*, Mathematische Zeitschrift 79 (1962), p. 21-31.

[12] R. Venkataraman, *Symmetric ordered groups*, ibidem 79 (1962) p. 10-20.


UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON

*Reçu par la Rédaction le 10. 4. 1967*