COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 2

CONCERNING POINT SETS WITH A SPECIAL CONNECTEDNESS PROPERTY

BY

B. E. WILDER (SECURITY, COLORADO)

1. Introduction. The purpose of this paper is to give a topological property which can be imposed on certain types of continua, or decompositions of them, to obtain "uniqueness" theorems for the resulting continuum or decomposition space. All point sets are assumed to be in a topological space satisfying Axioms 0 and 1_3 of [2]. Certain terminology and theorems from [2] are used without explicit mention.

Part of this work was accomplished while the author was at Auburn University. The example M of section 3 is due to Professor Ben Fitzpatrick, Jr.

2. Definitions and notations. If G is a collection of point sets, then G^* denotes the sum of all the elements of G. A *mapping* is a continuous transformation. An *irreducible continuum* is one which is irreducible between two of its points.

If each of M and N is a point set, then:

(i) M is said *to have property* C if and only if it is true that M has three points, and if X, Y and Z are three points of M, then M contains a continuum K which contains X and contains only one of the points Y and Z. M has property C *hereditarily* if and only if each non-degenerate continuum in M has property C. A continuum which has property C (hereditarily) will be called a *C-continuum* (*CH-continuum*).

(ii) The collection G of subsets of M is a *decomposition* of M if and only if G is an upper semi-continuous collection of mutually exclusive continua filling up M.

(iii) The decomposition G of M is *atomic* if and only if it is true that if K is a continuum in M intersecting two elements of G, then K is the sum of all the elements of G intersecting it.

(iv) A mapping f from M onto N is an *atomic mapping* if and only if it is true that if P is a point of N, then $f^{-1}(P)$ is connected and if K is a continuum in M such that $f(K)$ is non-degenerate, then $K = f^{-1}(f(K))$.

3. Property C. It is not difficult to show that each point set with property C is connected and that each arcwise connected point set has property C. Any indecomposable continuum would be a connected set without property C. A C-continuum which fails to be arcwise connected may be constructed as follows: let M_0 denote the closure in the plane of the simple graph whose x-projection is $(0, 1]$ and whose ordinate at the number x is $\sin(1/x)$, and let $M = M_0 \times [0, 1]$. Then if X, Y, and Z are three points of M, M contains a continuum K (which is either an arc, a copy of M_0, or the sum of an arc and a copy of M_0) containing X and containing only one of the points Y and Z. Thus property C is a generalization of arcwise connectedness which is not as general as connectedness.

4. Property C and arcs.

THEOREM 1. *The point set M is an arc if and only if it is a locally compact irreducible C-continuum.*

Proof. Every arc is a locally compact irreducible C-continuum. Suppose that M is a locally compact C-continuum which is irreducible between the points A and B. Let p be a point of $M - \{A, B\}$. Let H denote the subset of M to which r belongs if and only if every subcontinuum of M containing A and r contains p, and let K denote the subset of M to which r belongs if and only if every subcontinuum of M containing B and r contains p. Then B belongs to $H - H\cdot K$, A belongs to $K - H\cdot K$ and p belongs to $H\cdot K$.

Let r be a point of $M - \{A, B, p\}$. Then M contains a continuum K_1 containing A and containing only one of the points p and r, and a continuum K_2 containing B and containing only one of the points p and r. Now if either p or r belongs to both K_1 and K_2, the other point belongs to neither, and $K_1 + K_2$ is a proper subcontinuum of M containing A and B, which is impossible. Thus either p belongs to K_1 and r belongs to K_2 or vice versa.

Suppose first that p belongs to K_1. Then K_2 is a subcontinuum of M containing B and r but not p, and every subcontinuum of M containing A and r must contain p (or M is not irreducible from A to B). Similarly, if p belongs to K_2, then K_1 is a subcontinuum of M containing A and r but not p and every subcontinuum of M containing B and r contains p. Then r belongs to either H or K but not both. Moreover, if p belongs to K_1, K_2 is a connected subset of H containing B and r and if p belongs to K_2, K_1 is a connected subset of K containing A and r. Then letting $H' = H - p$ and $K' = K - p$, we have that H' and K' are mutually exclusive and connected and $M - p = H' + K'$.

Now suppose that q is a point of K' and is a limit point of H'. Then q is not B and some subcontinuum, D, of M contains B and contains only one of the points q and p. Since q is in K', D contains p and B but not q. Then there is a region, R, containing q which does not intersect D. R contains a point, r, of H'. Then r is not A, r is not in D, and some subcontinuum, E, of M contains A and contains only one of the points p and r. Since r is in H, E contains A and p but not r. Then $D+E$ is a subcontinuum of M containing A and B but not r. This contradicts the irreducibility of M from A to B. Thus no point of K' is a limit point of H'. Similarly, no point of H' is a limit point of K', and H' and K' are mutually separated.

Now suppose that $M-A$ is the sum of two mutually separated sets U and V, where V contains B. Let x and y be points of U. Then M contains a continuum W containing B and containing only one of the points x and y. Then W is a proper subcontinuum of M intersecting U and V so it contains A, which contradicts the irreducibility of M. Thus $M-A$ is connected. Similarly, $M-B$ is connected.

Thus A and B are non-cut points of the locally compact continuum M and if p is any other point of M, $M-p$ is the sum of two mutually separated connected point sets. Hence M is an arc from A to B.

COROLLARY 1. *A chainable continuum is a C-continuum if and only if it is an arc.*

Proof. Every chainable continuum is compact and irreducible. Thus a chainable C-continuum is an arc by Theorem 1. Every arc is a chainable C-continuum.

COROLLARY 2. *Every compact CH-continuum is arcwise connected.*

Remark. The analogy to Corollary 1 for circularly chainable continua does not hold true. Let M_0 be the plane continuum defined in section 3, and let K be an arc from $A = (0, -1)$ to $B = (1, \sin 1)$ having only the points A and B in common with M_0. Then $M = M_0+K$ is a circularly chainable C-continuum (it is arcwise connected) which is not a simple closed curve.

5. Property C, atomic decompositions and mappings. Anderson and Choquet [1], introduced the notion of an atomic mapping and stated that a continuum admitted a unique atomic decomposition into an arc or n—od. This section generalizes this result to atomic decompositions of point sets into sets with property C, and obtains a sufficient condition for an atomic mapping to be a topological transformation.

THEOREM 2. *If each of G and H is an atomic decomposition of the point set M and each of G and H, regarded as space, has property* C, *then $G = H$.*

Proof. For each subset A of M let $G(A)$ denote the collection of all the elements of G which intersect A. Suppose that some element, g, of G intersects two elements h_1 and h_2 of H. Let g' denote an element of G different from g. Then g' must intersect some element h' of H. Suppose $h' = h_1$. Then h_1 intersects the two elements g and g' of G, so $h_1 = G(h_1)^*$. Then g is a subset of h_1 and does not intersect h_2, which contradicts our original assumption. Thus $h' \neq h_1$. By a similar argument, $h' \neq h_2$. Since H has property C, H contains a continuum K containing h' and containing only one of the elements h_1 and h_2. Then K^* is a continuum in M intersecting the two elements g and g' of G, so $K^* = [G(K^*)]^*$. Then g is a subset of K^* and does not intersect both h_1 and h_2, which contradicts our original assumption. Thus each element of G intersects only one element of H. By a similar argument, each element of H intersects only one element of G. Hence $G = H$.

COROLLARY 3. *If M has property* C *and G is an atomic decomposition of M, then each element of G is degenerate.*

LEMMA. *If M is compact, N is non-degenerate, f is an atomic mapping from M onto N and G denotes the collection to which g belongs if and only if there is a point p of N such that $g = f^{-1}(p)$, then G is an atomic decomposition of M.*

THEOREM 3. *If M is compact and each non-degenerate component of M has property* C *and if f is an atomic mapping from M onto N such that if K is a non-degenerate component of M then $f(K)$ is non-degenerate, then f is a topological transformation.*

Proof. Let G denote the collection to which g belongs if and only if there is a point, p, of N such that $g = f^{-1}(p)$. Then G is an atomic decomposition of M. Suppose that some element g of G is non-degenerate. Then g is a subset of some component K of M, and $f(K)$ is non-degenerate so K intersects two elements of G. Let $G(K)$ denote the collection of all the elements of G intersecting K. Then $G(K)^* = K$, and $G(K)$ is an atomic decomposition of K. Since K has property C, each element of $G(K)$ is degenerate, which contradicts the assumption that g is non-degenerate. Thus each element of G is degenerate and f is reversible. Since M is compact, f is a topological transformation.

REFERENCES

[1] R. D. Anderson and G. Choquet, *A plane continuum no two of whose non-degenerate subcontinua are homeomorphic. An application of inverse limits*, Proceedings of the American Mathematical Society 10 (1959), p. 347-353.

[2] R. L. Moore, *Foundations of point set theory*, 1962.

Reçu par la Rédaction le 14. 9. 1967