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MINIMAL TOPOLOGIES

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It is well-known that every compact Hausdorff topology $\mathcal{T}$ on a set X is minimal in the sense that there is no Hausdorff topology on X properly contained in $\mathcal{T}$. Bourbaki [6] has characterized all spaces with this property and shown them to be absolutely closed; see [1]. He also exhibits a space of Urysohn which is minimal Hausdorff, but is not compact. A description of this space can be found in [4]. Banaschewski [3] has obtained analogous results for minimal regular spaces and Berri and Sorgenfrey [5] have exhibited a minimal regular space which is not compact. More recently Herrlich [7] has arrived at similar results for Urysohn spaces. The existence and characterization of these spaces were also known to C. T. Scarborough. Banaschewski [3] has investigated minimal locally compact and minimal completely regular spaces and shown them to be compact. In [4], Berri proves that every minimal normal space is compact.

We will investigate minimal λ-spaces where $\lambda =$ paracompact, metric, completely normal and completely Hausdorff, and show all these spaces to be compact. Only Hausdorff spaces will be considered.

The terminology will coincide with that of [3], [5] and [6]. The word "space" will mean topological space. If A is a subset of the space X, we denote the closure by A'.

1. Paracompact spaces. *A paracompact space* is a regular space in which every open cover has an open locally finite refinement. It is well-known [8] that in regular spaces this condition is equivalent to: every open cover has an open σ-locally finite refinement. We say that a paracompact space $(X, \mathcal{T})$ is *minimal paracompact* if there is no paracompact topology defined on X which is properly contained in $\mathcal{T}$.

THEOREM 1. *A minimal paracompact space* $(X, \mathcal{T})$ *is compact.*

It is well-known [2] that a pseudo-compact paracompact space is compact, so it suffices to show that every countable regular filter base

$\{F_n\}$ has an adherent point. Suppose that $\{F_n\}$ has void adherence. For each $x \in X$, let $\mathcal{N}(x)$ denote the open neighborhood system of x. Choose $p \in X$, and define a new fundamental system of neighborhoods on X as follows:

$$\mathcal{M}(x) = \mathcal{N}(x), \qquad x \neq p,$$

$$\mathcal{M}(x) = \{F_n \cup N: n = 1, 2, \ldots, N \in \mathcal{N}(p)\}, \qquad x = p.$$

Then $\mathcal{T}'$, the topology induced on X by $\{\mathcal{M}(x): x \in X\}$, is a regular Hausdorff topology which is strictly weaker than $\mathcal{T}$.

We will show that $\mathcal{T}'$ is a paracompact topology. Let $\mathcal{U}$ be an open cover of $(X, \mathcal{T}')$. Since $(X, \mathcal{T})$ is paracompact, there exists an open locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. There exist $N_1 \in \mathcal{N}(p)$ and F_n such that $N_1 \cup F_n$ is contained in some member $U \in \mathcal{U}$. By regularity of $\mathcal{T}$, there exists an $N_2 \in \mathcal{N}(p)$ such that $N_2' \subset N_1$. Let $\mathcal{W}_k = \{V - (F_k \cup N_2)': V \in \mathcal{V}\}$. It follows that $\mathcal{W} = \bigcup\{\mathcal{W}_k: k = 1, 2, \ldots\} \cup \{N_1 \cup F_n\}$ is an open σ-lo cally finite refinement of $\mathcal{U}$ with respect to $\mathcal{T}'$. Thus $\mathcal{T}'$ is paracompact, which contradicts the minimality of $\mathcal{T}$.

2. Metric spaces. A space $(X, \mathcal{T})$ is said to be *metrizable* if there exists a metric defined on X compatible with $\mathcal{T}$. We say that a space $(X, \mathcal{T})$ is *minimal metric* if $\mathcal{T}$ is metrizable and there is no metric topology defined on X which is properly contained in $\mathcal{T}$. If $(X, \mathcal{T})$ is metrizable and is closed in every metric space in which it is homeomorphically embedded, we say that $(X, \mathcal{T})$ is *metric closed.*

THEOREM 2. *A minimal metric space $(X, \mathcal{T})$ is compact.*

Since every metric space is paracompact, we may suppose as in Theorem 1 that there exists a countable regular filter base $\{F_n\}$ on $(X, \mathcal{T})$ having void adherence. Let d be a metric on X compatible with $\mathcal{T}$, and

$$N(x, 1/n) = \{y \in X: d(x, y) < 1/n\} \qquad \text{for each } x \in X.$$

Choose $p \in X$, and define a new fundamental system of neighborhoods on X as follows:

$$\mathcal{M}(x) = \{N(x, 1/n): n = 1, 2, \ldots\}, \qquad x \neq p,$$

$$\mathcal{M}(x) = \{F_n \cup N(x, 1/n): n = 1, 2, \ldots\}, \qquad x = p.$$

Then $\mathcal{T}'$, the topology induced on X by $\{\mathcal{M}(x): x \in X\}$, is a regular Hausdorff topology strictly weaker than $\mathcal{T}$.

We will show that $\mathcal{T}'$ is metrizable. By [9] and [10], it suffices to show that $\mathcal{T}'$ has a σ-locally finite base. Since $\mathcal{T}$ is metrizable, it has a σ-locally finite base $\mathcal{B} = \{\mathcal{B}_j: j = 1, 2, \ldots\}$. Let

$$\mathcal{W}_{nj} = \{B - (F_n \cup N(p, 1/n))': B \in \mathcal{B}_j\} \cup \{F_n \cup N(p, 1/n)\}$$

and

$$\mathscr{W} = \bigcup \{\mathscr{W}_{nj} : n, j = 1, 2, \ldots\}.$$

Then $\mathscr{W}$ is an open σ-locally finite family with respect to $\mathscr{T}'$.

It remains to show $\mathscr{W}$ is a base for $\mathscr{T}'$. Let $x \in U \in \mathscr{T}'$. If $x = p$, there exists an $F_n \cup N(p, 1/n) \subset U$. If $x \neq p$, then $x \in U - (F_n \cup N(p, 1/n))'$ for some n. Since $\mathscr{B}$ is a base for $\mathscr{T}$, there exists a $B \in \mathscr{B}_j$ for some j such that $x \in B \subset U$. Then

$$x \in B - (F_n \cup N(p, 1/n))' \subset U$$

and

$$B - (F_n \cup N(p, 1/n))' \in \mathscr{W}_{nj}.$$

Thus $\mathscr{W}$ is a base for $\mathscr{T}'$. It follows that $\mathscr{T}'$ is a metric topology, which contradicts the minimality of $\mathscr{T}$.

THEOREM 3. *Every metric closed space $(X, \mathscr{T})$ is compact.*

Let p be a point not in X, and let $Y = X \cup \{p\}$. Suppose there exists a countable regular filter $\{F_n\}$ on $(X, \mathscr{T})$ having void adherence. Using the notation of Theorem 2, we define a fundamental system of neighborhoods on Y as follows:

$$\mathscr{M}(x) = \mathscr{N}(x), \quad x \neq p,$$
$$\mathscr{M}(x) = \{F_n \cup \{p\} : n = 1, 2, \ldots\}, \quad x = p.$$

Then $\{\mathscr{M}(x) : x \in Y\}$ induces a regular Hausdorff topology $\mathscr{T}'$ on Y. By interchanging the symbol $F_n \cup \{p\}$ with $F_n \cup N(p, 1/n)$ of Theorem 2, it is easy to see that $(Y, \mathscr{T}')$ is metrizable. Thus $(X, \mathscr{T})$ is embedded as a non-closed subspace of the metrizable space $(Y, \mathscr{T}')$. This is a contradiction.

3. Completely normal spaces. A space $(X, \mathscr{T})$ is completely normal if for every pair of separated sets A and B contained in X, there exist open sets $U, V \in \mathscr{T}$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. A completely normal space $(X, \mathscr{T})$ is said to be *minimal completely normal* if there is no completely normal topology defined on X which is properly contained in $\mathscr{T}$. If $(X, \mathscr{T})$ is completely normal and is closed in every completely normal space in which it is embedded, we say that $(X, \mathscr{T})$ is *completely normal closed.*

THEOREM 4. *A minimal completely normal space $(X, \mathscr{T})$ is compact.*

Since $(X, \mathscr{T})$ is completely regular, it suffices to show that every regular filter base on $(X, \mathscr{T})$ has non-void adherence; see [5]. Suppose $\mathscr{F}$ is a regular filter base on $(X, \mathscr{T})$ with void adherence. Choose $p \in X$ and carry out the same procedure as in Theorem 1 to obtain a regular Hausdorff topology $\mathscr{T}'$ on X strictly weaker than $\mathscr{T}$.

We will show that $\mathscr{T}'$ is completely normal. Let A and B be two separated sets in $(X, \mathscr{T}')$. Then A and B are separated in $(X, \mathscr{T})$, so there exist sets U and V both members of $\mathscr{T}$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. If $p \notin A \cup B$, then $U - \{p\}$ and $V - \{p\}$ are both members of $\mathscr{T}'$ and $(U - \{p\}) \cap (V - \{p\}) = \emptyset$. Suppose $p \in A \cup B$, say $p \in A$. Then by regularity of $\mathscr{T}'$, there exist sets S, T in $\mathscr{T}'$ such that $p \in S, B \subset T$ and $S \cap T = \emptyset$. It follows that $A \subset S \cup U \in \mathscr{T}'$, $B \subset T \cap V \in \mathscr{T}'$, and $(S \cup U) \cap (T \cap V) = \emptyset$. Thus $(X, \mathscr{T}')$ is completely normal. This is a contradiction.

THEOREM 5. *A completely normal closed space* $(X, \mathscr{T})$ *is compact.*

As in the previous theorem, we suppose $\mathscr{F}$ is a regular filter base with void adherence. Let p be a point not in X and let $Y = X \cup \{p\}$. Using the method of Theorem 3, we define a completely normal topology $\mathscr{T}'$ on Y. Thus $(X, \mathscr{T})$ is embedded as a non-closed subset of the completely normal space $(Y, \mathscr{T}')$. This is a contradiction.

4. Completely Hausdorff spaces. A space X is *completely Hausdorff* if for every pair of distinct points x, y in X, there exists a real-valued continuous function f on X such that $f(x) \neq f(y)$. We say that a completely Hausdorff space $(X, \mathscr{T})$ is *minimal completely Hausdorff* if there is no completely Hausdorff topology properly contained in $\mathscr{T}$. If $(X, \mathscr{T})$ is completely Hausdorff and is closed in every completely Hausdorff space in which it is embedded, we say that $(X, \mathscr{T})$ is *completely Hausdorff closed.*

THEOREM 5. *A minimal completely Hausdorff space* $(X, \mathscr{T})$ *is compact.*

First we observe that $(X, \mathscr{T})$ is minimal completely Hausdorff if and only if every one-to-one continuous function from $(X, \mathscr{T})$ onto a completely Hausdorff space is a homeomorphism. Let F be the set of all continuous functions from $(X, \mathscr{T})$ to $[0, 1]$. Define $g: X \to [0, 1]^F$ such that $g(x)_f = f(x)$. Then g is continuous since each $f \in F$ is continuous. If $x \neq y$, there exists on $f \in F$ such that $f(x) \neq f(y)$, so g is one-to-one. Hence g is a homeomorphism, and $(X, \mathscr{T})$ is completely regular. Since $(X, \mathscr{T})$ is minimal completely Hausdorff, it is minimal completely regular; by [3], $(X, \mathscr{T})$ is compact.

Using the method of Theorem 5, we see that completely Hausdorff implies compactness in minimal Urysohn and minimal Hausdorff spaces; see [7], p. 290.

It is not true that a completely Hausdorff closed space is compact, even if it is absolutely closed.

Example. Let X consist of the points a_{ij}, c_i, and a, where $i, j = 1, 2, \ldots$, in the notation of [4]. It is easy to see that X is absolutely closed and completely Hausdorff, but not compact.

Finally we observe that the proofs of Theorems 1, 2, and 4 give methods of constructing strictly weaker paracompact, metric and completely normal topologies from non-compact topologies of the same type.

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