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ON THE PLANE ONE-TO-ONE MAP OF A LINE

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Lelek and McAuley [1] have shown that if the metric image X of the line E^1 under a one-to-one continuous mapping is both locally connected and locally compact, then X is homeomorphic to one of five rather simple curves: an open line interval, a figure eight, a dumb-bell curve, a theta curve, or a noose (circle plus open interval). Since all of these curves are embeddable in the plane, they question the necessity of the local compactness part of the hypothesis if the mapping is assumed to be into the plane (P 615 of [1]). The purpose of this note is to indicate the proof that for 1-1 plane maps local compactness is a consequence of (but not equivalent to) local connectedness.

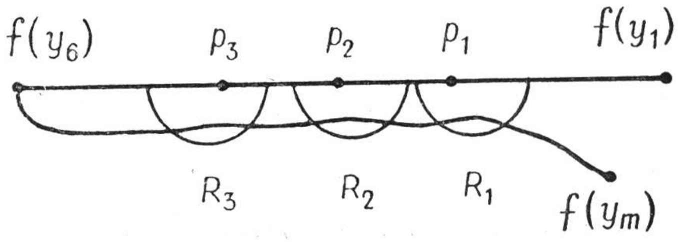
THEOREM. *Suppose that X is the range of a continuous, one-to-one, function from the line E^1 to the plane E^2. If X is locally connected, then X is a locally compact subspace of E^2.*

Indication of proof (1). Suppose, on the contrary, that the function $f\colon E^1 \to E^2$ is a 1-1 and continuous such that $X = f(E^1)$ is a locally connected subspace of E^2, which is not locally compact. Then there exist two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of real numbers and distinct points p of X and q of E^2 (q possibly is ∞) such that $f(x_i) \to p$ and $f(y_i) \to q$. Without loss of generality we shall assume that $x_i \to \infty$, $y_i \to \infty$ and $y_i < x_i < y_{i+1}$.

Let D denote a connected, relatively open, subset of X which contains p such that $q \in E^2 - \bar{D}$. For each i, suppose that $p_i = f(x_i) \in D$ and $f(y_i) \in E^2 - \bar{D}$; let T_i denote the component of $D \cap f([y_i, y_{i+1}])$ which contains p_i. Obviously T_i contains a limit point of $D - T_i$ and of the first five some three of them, say the first three, must contain limit points from the same side. (Since all five are subsets of the arc $A = f([y_1, y_6])$, this makes sense. For definitions and theorems about sides of arcs the reader is referred to [2], p. 180-185.) With a homeomorphism of the

(1) [*Note of the Editors.*] Answering an inquiry of the Editors, the author has explained in a letter of April 2, 1968, that he shall work on the proof again and hopes to submit it after completion to Fundamenta Mathematicae.

plane onto itself $f([y_1, y_6])$ may be embedded in the x-axis (of the plane); so we lose no generality in assuming that the situation is as pictured in figure, where R_1, R_2 and R_3 are semi-circular plane regions such that



$R_1 \cap X$, $R_2 \cap X$ and $R_3 \cap X$ are non-void subsets of D and contain p_1, p_2 and p_3 in their respective boundaries relative to X. Let D_i $(i = 1, 2, 3)$ denote a component of $R_i \cap D$ whose closure contains a point of A. There exist a natural number m and a subarc B of $f([y_6, y_m])$ such that the arc B intersects D_1, D_2 and D_3 and $A \cap B = f(y_6)$. With the proper selection of m and B, D_1, D_2 and D_3 may be taken to be components of $R_1 \cap (D-B)$, $R_2 \cap (D-B)$ and $R_3 \cap (D-B)$, respectively, each of which has limit points in both A and B (i.e., $\bar{D}_i \cap A \neq \emptyset \neq \bar{D}_i \cap B$ for $i = 1, 2, 3$) such that no one of them has limit points in $A \cup B$ from both sides.

Two cases arise but in either case D_2 cuts D_1 from D_3 in $E^2-(A \cup B)$ (i.e., every continuum in $E^2-(A \cup B)$ which intersects both D_1 and D_3 must intersect D_2). It follows with the help of Theorem 34, p. 182, of [2] that for $i \neq j$ no two points of $\bar{D}_i \cap (A \cup B)$ separate two points of $\bar{D}_j \cap (A \cup B)$ on $A \cup B$ considering $A \cup B$ to be a simple closed curve with $f(y_1) = f(y_m)$. Hence there exist arbitrarily large natural numbers n such that $f([y_n, y_{n+1}])$ intersects both D_1 and D_3 and some component of $f([y_n, y_{n+1}]) \cap D_2$ must contain limit points of D_2 from both sides. This establishes the useful lemma that this property being possessed by D must also be possessed by D_1 and D_3 and in fact, by any small connected open subset of X whose closure is not compact (in X).

Now let us take the case where the boundary of R_1 contains an arc C such that $A \cup B \cup C$ contains a simple closed curve J whose interior I contains $D_2 \cup D_3$. There are only countably many open arcs T such that for some $n \geqslant m$, T is a component of $f([y_n, y_{n+1})] \cap I$ which intersects D_3 and contains limit points of D_3 from both sides. Let α denote a simple well-ordering of all such open arcs T. Assign a 1-side and a 0-side to each arc of α and to each diadic decimal $.n_1 n_2 n_3 \ldots$ $(n_i = 0, 1)$ let K denote a continuum obtained as follows:

Let T_1 denote the first term of α; T_1 separates I into two disjoint simple domains. If n_1 is 0, let I_1 denote that one of these two which contains points of D_3 on the 0-side of T_1; otherwise, if n_1 is 1, let I_1 denote the one which contains points of D_3 on the 1-side of T_1. Let T_2 denote the first term of a which lies in I_1; T_2 separates I_1 into two disjoint domains. Again if n_2 is 0, let I_2 denote the one of these two which contains points of D_3 on the 0-side of T_2; otherwise let I_2 denote the other one. Now let K denote $\bigcap \bar{I}_i$ and let $\{K\}$ denote the collection of all such continua. There is at least one member of $\{K\}$ for each number in $[0, 1]$ and no two of them have in common more than one arc in a. Except for one member of $\{K\}$ (the one that contains $J-C$), each member of $\{K\}$ separates D_2. (Theorem 28, p. 156 of [2].) Only countably many components of $f([a, b]) \cap I$, where $f(a) \notin I$ and $f(b) \notin I$, intersect D_2. If every point of D_2 belongs to such a component, then some member of $\{K\}$ contains no point of D_2, which is a contradiction. If some point of D_2 fails to belong to such a component, then only minor changes in the construction are required to avoid this case.

EXAMPLES. In [1] Lelek and McAuley give an example of a 1-1 continuous image of the line into 3-space E^3 which is locally connected without being a locally compact subspace. Another way to produce such sets is as follows: Let B denote a countable basis for E^3 of spherical regions and let β denote a simple well-ordering of all disjoint pairs of elements of B. Draw a ray so that it runs straight between some two points a_1 and b_1 such that a_1 belongs to one member of the first term β_1 of β and b_1 belongs to the other member of β_1. Then continue to draw the ray (in a continuous fashion) so that again it runs straight between points of the members of β_2 respectively, etc. In E^3 this can always be done so the ray never crosses itself (i.e., multiple points may be avoided) and for each n, some arc of it runs straight between the members of β_n. The resulting set will be connected and locally connected.

REFERENCES

[1] A. Lelek and L. F. McAuley, *On hereditarily locally connected spaces and one-to-one continuous images of a line*, Colloquium Mathematicum 17 (1967), p. 319-324.

[2] R. L. Moore, *Foundations of point set theory*, Providence 1962.

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