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PSEUDOREGULAR MAPPINGS

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1. Introduction. Let P be a local property of topological spaces which is preserved under finite products. A bundle map f from a space X onto a space Y satisfies the following: If Y and each point inverse under f have property P, then X has property P. In this paper* a pseudoregular map will be defined and the above phenomenon will be investigated, where it will be assumed that f is pseudoregular. The main theorems of this paper are:

(I) If f is a pseudoregular map of X onto Y, and Y and each point inverse under f are Peano continua, then X is a Peano continuum.

(II) If f is a closed pseudoregular map of X onto Y, and Y and each point inverse under f are locally compact, then X is locally compact.

It would be very interesting if a similar theorem could be proved if Y and each point inverse under f are locally connected in dimension n.

Dyer and Hamstrom [1] defined a completely regular map (of which pseudoregular is a generalization) and showed that under certain conditions completely regular maps are bundle maps. It would be then of interest to find connexions between pseudoregular mappings and fiber maps of some type.

All spaces will be considered Hausdorff and the notation $f\colon X \to\to Y$ will mean that f is a continuous function (map) of X onto Y. The notation $S(\varepsilon, x)$ will mean an ε-neighborhood of x and Cl will be used to denote closure.

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2. Pseudoregular mappings. The following definition is due to Dyer and Hamstrom [1]:

(2.1) Definition. A mapping f of a metric space X onto a metric space Y is said to be *completely regular* provided that it is true that for

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each point y of Y and each positive number ε there is a positive number δ such that if x is a point of Y and $d(x, y) < \delta$, then there exists a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(x)$ which moves no point as much as ε. (Such a homeomorphism will be called an *ε-homeomorphism*).

The following definition is a generalization of (2.1):

(2.2) Definition. A mapping f of a metric space X onto a metric space Y is said to be *pseudoregular* provided that it is true that for each point y of Y and each positive number ε there is a positive number δ such that if x is a point of Y and $d(x, y) < \delta$, then there is a continuous map of $f^{-1}(y)$ onto $f^{-1}(x)$ which moves no point as much as ε. (Such a map will be called an *ε-map*.)

(2.3) THEOREM. *If f is a pseudoregular mapping from a metric space (X, d) onto a metric space (Y, ϱ), then f is an open mapping.*

Proof. Let U be an open set in X and let $x \in U$. There exists an $\varepsilon > 0$ such that $S(\varepsilon, x) \subset U$ and there exists a $\delta > 0$ such that if $\varrho(y, f(x)) < \delta$, then there exists an ε-map $h_y: f^{-1}f(x) \to\to f^{-1}(y)$. It will be shown that

$$S(\delta, f(x)) \subset f(S(\varepsilon, x)) \subset f(U)$$

and hence the proof will be complete. Let $y \in S(\delta, f(x))$; then there exists an ε-map $h_y: f^{-1}f(x) \to\to f^{-1}(y)$. Therefore $h_y(x) \in S(\varepsilon, x)$ and $y = f(h_y(x)) \subset f(S(\varepsilon, x))$. Hence $S(\delta, f(x)) \subset f(S(\varepsilon, x))$ as desired.

(2.4) THEOREM. *Let f be a pseudoregular mapping from a metric space (X, d) onto a metric space (Y, ϱ). If Y and $f^{-1}(y)$ are locally connected for all y in Y, then X is locally connected.*

Proof. Let U be an open set in X, let C be a component of U and let $x \in C$. It will be shown that C is a neighborhood of x. Since $f^{-1}f(x)$ is locally connected, there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a neighborhood V of x such that

$$x \in S(4\varepsilon_2, x) \subset V \subset S(\varepsilon_1, x) \subset S(4\varepsilon_1, x) \subset U$$

and $V \cap f^{-1}f(x)$ is connected. Since f is pseudoregular, there exists a $\delta > 0$ such that if $\varrho(y, f(x)) < \delta$, then there exists an ε_2-map $h_y: f^{-1}f(x) \to\to f^{-1}(y)$. And since Y is locally connected, there exists an open connected set Z such that $f(x) \in Z \subset S(\delta, f(x))$.

Let $W = \bigcup_{y \in Z} h_y(f^{-1}f(x) \cap V)$.

(a) $h_y(f^{-1}f(x) \cap V)$ is connected and contained in $f^{-1}(y)$ trivially.

(b) $f(W) = Z$ trivially.

(c) W is a neighborhood of x by the following argument. There exists $\delta_1 > 0$ such that $S(\delta_1, f(x)) \subset Z$ and there exists $0 < \varepsilon_3 < \varepsilon_2$ such that if $d(x, w) < \varepsilon_3$, then $\varrho\big(f(x), f(w)\big) < \delta$. It will now be shown that $S(\varepsilon_3, x) \subset W$. To prove this, let $w \in S(\varepsilon_3, x)$; then there exists an ε_2-map $h_{f(w)}: f^{-1}f(x) \to\to f^{-1}f(w)$ and hence there exists $u \in f^{-1}f(x)$ such that

$$h_{f(w)}(u) = w, \quad d(x, u) \leqslant d(x, w) + d(w, u) < \varepsilon_3 + \varepsilon_2 < 4\varepsilon_2.$$

That is

$$w \in h_{f(w)}\big(f^{-1}f(x) \cap S(4\varepsilon_2, x)\big) \subset h_{f(w)}\big(f^{-1}f(x) \cap V\big) \subset W.$$

Therefore $S(\varepsilon_3, x) \subset W$ as desired;

(d) W is connected. This follows from the following argument. Assume that W is not connected. Then $W = O \cup P$, O and P open in W, and $O \cap P = \emptyset$. Then

$$O = \bigcup_{y \in S} h_y\big(f^{-1}f(x) \cap V\big) \quad \text{and} \quad P = \bigcup_{y \in T} h_y\big(f^{-1}f(x) \cap V\big),$$

where $S = f(O)$ and $T = f(P)$. Then $S \cup T = Z$ since $Z = f(W) = f(O) \cup f(P) = S \cup T$, and S and T are non-empty. Since Z is connected, either $\bar{S} \cap T \neq \emptyset$ or $\bar{T} \cap S \neq \emptyset$. Assume $\bar{S} \cap T \neq \emptyset$. Then there exists a sequence $\{s_i \mid s_i \in S\}$ such that $\lim s_i = t$ where $t \in T$. Since f is an open map,

$$\liminf f^{-1}(s_i) \supset f^{-1}(t).$$

Hence there is a sequence $\{v_i \mid v_i \in f^{-1}(s_i)\}$ which converges to $h_t(x) \in f^{-1}(t)$. Assume that $v_i = h_{f(s_i)}(u_i)$, where $u_i \in f^{-1}f(x)$. By the above there exists a positive integer N such that if $n > N$, then $d\big(v_n, h_t(x)\big) < \varepsilon_2$. It will now be shown that if $n > N$, then $u_n \in V$. Since if $n > N$,

$$d(u_n, x) \leqslant d(u_n, v_n) + d\big(v_n, h_t(x)\big) + d\big(h_t(x), x\big) < \varepsilon_2 + \varepsilon_2 + \varepsilon_2 < 4\varepsilon_2.$$

Therefore $u_n \in S(4\varepsilon_2, x) \subset V$ and hence $v_n = h_{s_n}(u_n) \in h_{s_n}\big(f^{-1}f(x) \cap V\big) \subset W$.

This yields a contradiction since we now have a sequence $\{v_n\}$ which converges to $h_t(x)$ and $v_n \in O$, $h_t(x) \in P$. Therefore W is connected.

(e) $W \subset U$ since if $w \in W$, then $f(w) \in Z$ and hence there exists a $y \in f^{-1}f(x) \cap V$ such that $h_{f(w)}(y) = w$. Therefore

$$d(x, w) \leqslant d(x, y) + d(y, w) < \varepsilon_1 + \varepsilon_2 < 4\varepsilon_1$$

and hence $w \in S(4\varepsilon_1, x) \subset U$.

(f) C is a neighborhood of x since W is a connected neighborhood of x and hence $W \subset C$.

This completes the proof, since we have shown that, given any open set U of X and any component C of U, then C is open.

(2.5) Lemma. *A mapping f from a topological space X onto a topological space Y is closed if and only if given any y in Y and any neighborhood U of $f^{-1}(y)$ then there exists a neighborhood V of y such that if $v \in V$, then $f^{-1}(v) \subset U$ (i.e., $f^{-1}(V) \subset U$).*

Proof. This is just a restatement of the theorem that "A decomposition $\mathscr{D}$ of a topological space X is upper semi-continuous if and only if the projection P of X onto $\mathscr{D}$ is closed" ([2], Theorem 12, p. 99).

(2.6) LEMMA. *A pseudoregular map such that the inverse of each point is compact is a closed map.*

Proof. Let $f\colon X \to\to Y$ be a pseudoregular map of X onto Y such that $f^{-1}(y)$ is compact for all y in Y. Let $y \in Y$ and let U be a neighborhood of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there exists an $\varepsilon > 0$ such that $S(\varepsilon, f^{-1}(y)) \subset U$ and since f is pseudoregular, there exists a positive number δ such that if $\varrho(y, w) < \delta$, then there is an ε-map $h_w\colon f^{-1}(y) \to\to f^{-1}(w)$. Therefore if $w \in S(\delta, y)$, then $f^{-1}(w) \subset S(\varepsilon, f^{-1}(y))$ and hence f is closed by (2.5).

(2.7) Definition. A mapping f is *quasi-compact* if for every open set of the form $U = f^{-1}f(U)$, $f(U)$ is open.

(2.8) LEMMA. *If f is a quasi-compact mapping of a topological space X onto a connected space Y such that $f^{-1}(y)$ is connected for all y in Y, then X is connected.*

The proof is trivial.

It should be noted that if f is an open or closed mapping, then f is quasi-compact. Hence, by (2.3), any pseudoregular mapping is quasi-compact. The following lemma is due to Ponomarev [3]:

(2.9) LEMMA. *If $f\colon X \to\to Y$ is a closed mapping such that Y and $f^{-1}(y)$ are compact for all y in Y, then X is compact.*

(2.10) THEOREM. *If f is a pseudoregular mapping from a metric space X onto a Peano continuum Y such that $f^{-1}(y)$ is a Peano continuum for all y in Y, then X is a Peano continuum.*

The proof follows from (2.4), (2.6), (2.8), (2.9) and the Hahn-Mazurkiewicz Theorem [5].

(2.11) THEOREM. *Let f be a closed pseudoregular map of a metric space X onto a metric space Y such that Y and $f^{-1}(y)$ are locally compact for all y in Y. Then X is locally compact.*

Proof. Let $x \in X$, let U be a compact neighborhood of $f(x)$ and let V be a closed neighborhood of x such that $V \cap f^{-1}f(x)$ is compact. There exists an $\varepsilon > 0$ such that $S(4\varepsilon, x) \subset V \cap f^{-1}(U)$ and by the pseudoregularity there exists a positive number δ such that, if $\varrho(y, f(x)) \leqslant \delta$, there exists an ε-map h_y of $f^{-1}f(x)$ onto $f^{-1}(y)$. δ could also be chosen such that $S(\delta, f(x)) \subset U$. By the continuity of f there exists a positive number $\varepsilon_1 < \varepsilon$ such that if $d(x, w) < \varepsilon_1$, then $\varrho(f(x), f(w)) < \delta$. Let

$$C = \bigcup_{y \in \mathrm{Cl}[S(\delta, f(x))]} h_y(f^{-1}f(x) \cap V).$$

It will be shown that $S(\varepsilon_1, x) \subset C$.

Let $w \epsilon S(\varepsilon_1, x)$; then $\varrho\big(f(x), f(w)\big) < \delta$ and hence there exists an ε-map $h_{f(w)}$ of $f^{-1}f(x)$ onto $f^{-1}f(w)$. We now want to show that $w \epsilon h_{f(w)}\big(f^{-1}f(x) \cap V\big)$ or that there exists a $v \epsilon f^{-1}f(x) \cap V$ such that $h_{f(w)}(v) = w$. This follows since let v be any element of $f^{-1}f(x)$ such that $h_{f(w)}(v) = w$,

$$d(x, v) \leqslant d(x, w) + d(w, v) < \varepsilon_1 + \varepsilon < 2\varepsilon.$$

Therefore $v \epsilon S(2\varepsilon, x) \subset V$ and hence $S(\varepsilon_1, x) \subset C$.

Therefore $\mathrm{Cl}[S(\frac{1}{2}\varepsilon_1, x)] \subset S(\varepsilon_1, x) \subset C$. Let $g = f|\mathrm{Cl}[S(\frac{1}{2}\varepsilon_1, x)]$; g is a closed map. $g\{\mathrm{Cl}[S(\frac{1}{2}\varepsilon_1, x)]\}$ is a closed subset of U and hence is compact. $g^{-1}(y)$ is closed and contained in $h_y\big(f^{-1}f(x) \cap V\big)$ and hence $g^{-1}(y)$ is compact. Therefore, by (2.9), $\mathrm{Cl}[S(\frac{1}{2}\varepsilon_1, x)]$ is compact and hence X is locally compact.

(2.13) Definition. A locally compact connected space will be called a *generalized continuum* [5].

(2.14) THEOREM. *If f is a closed pseudoregular map of a metric space (X, d) onto a metric space (Y, ϱ) such that Y and $f^{-1}(y)$ are generalized continua for all y in Y, then X is a generalized continuum.*

The proof easily follows from (2.8) and (2.11).

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