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CHERN CLASSES AND BOCHNER-KAEHLER METRICS

BY

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1. Introduction. Let M be a compact complex manifold with complex structure J. Let g be a Hermitian metric on M. The fundamental 2-form Φ of g is defined by

$$\Phi(X, Y) = \tfrac{1}{2}g(JX, Y) \quad \text{for } X, Y \in TM,$$

where TM is the tangent bundle of M. The Hermitian metric g is called a *Kaehler metric* if $d\Phi = 0$. For a Kaehler metric g on M, the cohomology class ω represented by Φ is called the *fundamental class* of g.

For any Kaehler metric g on M and any closed (1, 1)-form ψ on M such that $\Phi+\psi$ is positive, $\Phi+\psi$ induces another Kaehler metric $\tilde{g}$ on M with $\Phi+\psi$ as its fundamental 2-form. In particular, if ψ is exact, the Kaehler metrics g and $\tilde{g}$ have the same fundamental class. This fact shows that there are many distinct Kaehler metrics on M with a preassigned fundamental class. In the following, two Kaehler metrics on M are said to be *cohomologous* if they have the same fundamental class. By a *complex space form* $M^n(c)$ we mean a Kaehler manifold with constant holomorphic sectional curvature c.

In the following, a Kaehler metric g on a complex manifold is called a *BK-metric* if its Bochner tensor vanishes. In [2] the first-named author is able to classify all compact analytic complex surfaces which admit BK-metrics. In [3], he proves that a compact complex manifold with vanishing first Chern class admits a BK-metric if and only if it is covered biholomorphically by C^n.

In this paper* we shall prove the following results:

THEOREM 1. *Let M be a compact complex manifold which admits a BK-metric $\tilde{g}$. If g is any Kaehler metric on M with constant scalar curvature such that g and $\tilde{g}$ are cohomologous, then (M, g) and $(M, \tilde{g})$ are equivalent. In particular, g is also a BK-metric and $\tilde{g}$ has constant scalar curvature.*

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THEOREM 2. *A compact complex manifold M is biholomorphically equivalent to a complex projective space CP^n if and only if M has positive first Chern class and it admits a BK-metric.*

THEOREM 3. *A compact complex manifold M is covered biholomorphically by the unit ball D^n of C^n if and only if it admits a BK-metric g such that the first Chern class c_1 satisfies $c_1 = a\omega$ for some constant $a < 0$, where ω denotes the fundamental class of g.*

2. Basic formulas. Let M be a Kaehler manifold of n complex dimensions. Let $\theta^1, \ldots, \theta^n$ be a local field of unitary coframes. Then the Kaehler metric is written as

$$g = \frac{1}{2} \sum (\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i)$$

and the fundamental 2-form is given by

$$\Phi = \frac{\sqrt{-1}}{2} \sum \theta^i \wedge \bar{\theta}^i$$

which is a harmonic form. Let

$$\Omega^i_j = \sum R^i_{jk\bar{l}}\, \theta^k \wedge \bar{\theta}^l$$

be the curvature form of M. Then the curvature tensor of M is the tensor field with local components $R^i_{jk\bar{l}}$, which will be denoted by R. The Ricci tensor S and the scalar curvature ϱ are given by

$$S = \frac{1}{2} \sum (R_{i\bar{j}}\, \theta^i \otimes \bar{\theta}^j + \bar{R}_{i\bar{j}}\, \bar{\theta}^i \otimes \theta^j) \quad \text{and} \quad \varrho = 2 \sum R_{i\bar{i}},$$

where $R_{i\bar{j}} = 2\sum R^k_{ik\bar{j}}$. The Bochner tensor B is a tensor field with local components

$$(2.1) \qquad B^i_{jk\bar{l}} = R^i_{jk\bar{l}} - \frac{1}{2(n+2)} (R_{\bar{i}k}\, \delta_{jl} + R_{\bar{i}j}\, \delta_{kl} + \delta_{ik} R_{j\bar{l}} + \delta_{ij} R_{k\bar{l}}) + \\ + \frac{\varrho}{4(n-1)(n-2)} (\delta_{ik}\, \delta_{jl} + \delta_{ij}\, \delta_{kl}).$$

We denote by $\|R\|$, $\|S\|$ and $\|B\|$ the length of the curvature tensor, the Ricci tensor and the Bochner tensor, respectively, so that

$$\|R\|^2 = 16 \sum R^i_{jk\bar{l}} R^j_{il\bar{k}}, \quad \|S\|^2 = 2 \sum R_{i\bar{j}} R_{j\bar{i}}, \quad \|B\|^2 = 16 \sum B^i_{jk\bar{l}} B^j_{il\bar{k}}.$$

It is easily seen that

$$(2.2) \qquad \|B\|^2 = \|R\|^2 - \frac{8}{n+2} S^2 + \frac{2}{(n+1)(n+2)} \varrho^2.$$

If $\tilde{g}$ is another Kaehler metric on the manifold M, we shall denote the corresponding quantities associated with $\tilde{g}$ by adding $\tilde{}$ on the top of them. For example, we denote by dV and $d\tilde{V}$ the volume elements of g and $\tilde{g}$, respectively.

3. Proof of Theorem 1. We first prove two lemmas.

LEMMA 1. *Let $\tilde{g}$ be a BK-metric on a compact complex manifold M and g any Kaehler metric on M which is cohomologous to $\tilde{g}$. If*

$$\int \varrho^2 dV \leqslant \int \tilde{\varrho}^2 d\tilde{V}, \tag{3.1}$$

then $\int \varrho^2 dV = \int \tilde{\varrho}^2 d\tilde{V}$ and g is also a BK-metric.

Proof. Since g and $\tilde{g}$ are cohomologous, their fundamental classes ω and $\tilde{\omega}$ are the same. In particular, the first and second Chern classes satisfy

$$\omega^{n-2}c_1^2 = \tilde{\omega}^{n-2}c_1^2 \quad \text{and} \quad \omega^{n-2}c_2 = \tilde{\omega}^{n-2}c_2, \tag{3.2}$$

where n denotes the complex dimensions of M.

Let γ_1 and γ_2 be given by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\sum \Omega_i^i, \tag{3.3}$$

$$\gamma_2 = \frac{1}{8\pi^2}\sum (\Omega_j^i \wedge \Omega_i^j - \Omega_i^i \wedge \Omega_j^j). \tag{3.4}$$

Then c_1 and c_2 are represented by γ_1 and γ_2, respectively.

From (3.2), (3.3) and (3.4) we see that there exist two $(2n-1)$-forms γ_1 and γ_2 such that

$$\Phi^{n-1}\wedge \gamma_1^2 = \tilde{\Phi}^{n-2}\wedge\tilde{\gamma}_1^2 + d\eta_1, \tag{3.5}$$

$$\Phi^{n-2}\wedge \gamma_2 = \tilde{\Phi}^{n-2}\wedge\tilde{\gamma}_2 + d\eta_2. \tag{3.6}$$

On the other hand, by direct computation we have

$$\Phi^{n-2}\wedge\gamma_1^2 = \frac{1}{16n(n-1)\pi^2}(\varrho^2 - 2\|S\|^2)\Phi^n, \tag{3.7}$$

$$\Phi^{n-2}\wedge\gamma_2 = \frac{1}{32n(n-1)\pi^2}(\varrho^2 - 4\|S\|^2 + \|B\|^2)\Phi^n. \tag{3.8}$$

Therefore, using (3.5), (3.6) and Stokes' theorem we obtain

$$\int(\varrho^2 - 2\|S\|^2)dV = \int(\tilde{\varrho}^2 - 2\|\tilde{S}\|^2)d\tilde{V}, \tag{3.9}$$

$$\int(\varrho^2 - 4\|S\|^2 + \|B\|^2)\,dV = \int(\tilde{\varrho}^2 - 4\|\tilde{S}\|^2 + \|\tilde{B}\|^2)d\tilde{V}. \tag{3.10}$$

Hence we get

$$\int\left(\|R\|^2-\frac{8}{n+2}\|S\|^2+\frac{2}{(n+1)(n+2)}\varrho^2\right)dV \tag{3.11}$$
$$=\int\left(\|\tilde{R}\|^2-\frac{8}{n+2}\|\tilde{S}\|^2-\frac{n-2}{n+2}\tilde{\varrho}^2\right)d\tilde{V}+\frac{n^2-n}{(n+1)(n+2)}\int\varrho^2 dV.$$

By the assumption that $\tilde{g}$ is a BK-metric, (2.2) and (3.11) imply

$$\int\|B\|^2 dV=\frac{n^2-n}{(n+1)(n+2)}\left(\int\varrho^2 dV-\int\tilde{\varrho}^2 d\tilde{V}\right),$$

whence we see that if (3.1) holds, then g is also a BK-metric and $\int\varrho^2 dV = \int\tilde{\varrho}^2 d\tilde{V}$. This completes the proof of the lemma.

LEMMA 2. *Let g and $\tilde{g}$ be two cohomologous Kaehler metrics on a compact complex manifold M. If g has constant scalar curvature ϱ, then*

$$\int\varrho^2 dV\leqslant\int\tilde{\varrho}^2 d\tilde{V}.$$

The equality sign holds when and only when $\tilde{g}$ has constant scalar curvature.

Proof. Since the Kaehler metrics g and $\tilde{g}$ are cohomologous, we have

$$\omega^n=\tilde{\omega}^n \quad \text{and} \quad \omega^{n-1}c_1=\tilde{\omega}^{n-1}c_1. \tag{3.12}$$

From the first equality we have

$$\int dV=\int d\tilde{V}. \tag{3.13}$$

From the second equality of (3.12) and (3.13) we find

$$\int\varrho dV=\int\tilde{\varrho}d\tilde{V}. \tag{3.14}$$

Combining (3.13), (3.14) and the Schwartz inequality, we have

$$\left(\int dV\right)\left(\int\varrho^2 dV\right)=\left(\int\varrho dV\right)^2=\left(\int\tilde{\varrho}d\tilde{V}\right)^2\leqslant\left(\int dV\right)\left(\int\tilde{\varrho}^2 d\tilde{V}\right), \tag{3.15}$$

which implies the lemma, since the inequality in (3.15) becomes equality when and only when $\tilde{\varrho}$ is constant.

From Lemmas 1 and 2 we see that both g and $\tilde{g}$ are BK-metrics and they have the same scalar curvature. By a result of [7], this implies that both (M, g) and $(M, \tilde{g})$ are either complex space forms or they are locally a product of two Kaehler manifolds of constant holomorphic sectional curvature H $(H>0)$ and $-H$, respectively. Thus, M is covered biholomorphically by CP^n, C^n, D^n or $CP^m\times D^{n-m}$. Since g and $\tilde{g}$ are cohomologous and (M, g) and $(M, \tilde{g})$ are Kaehler manifolds of above types, (M, g) and $(M, \tilde{g})$ must be equivalent. This proves Theorem 1.

4. Proof of Theorems 2 and 3. Let M be a compact complex manifold of n complex dimensions. If $n = 1$, every Kaehler metric on M is a BK-metric. Thus Theorems 2 and 3 are reduced to some well-known results.

Now we assume that $n > 1$ and M admits a BK-metric g with fundamental class ω.

If the first Chern class c_1 of M is positive, there is a $(1, 1)$-form $\bar{\gamma}$ on M which represents c_1 and $\bar{\gamma}$ is positive definite. By a well-known result of Yau [10], there is a unique Kaehler metric $\bar{g}$ whose Ricci form is $\bar{\gamma}$ and whose fundamental form is ω. Since $\bar{g}$ has positive definite Ricci form, a result of Kobayashi [5] shows that M is simply connected. Moreover, a compact Kaehler manifold with positive definite Ricci form being Hodge (see [5]), $\bar{g}$ is a Hodge metric. Since g and $\bar{g}$ are cohomologous, g is thus also a Hodge metric. Theorem 2.2 of [8] then implies that (M, g) is holomorphically isometric to the complex projective space CP^n with a standard Fubini-Study metric.

If M admits a BK-metric g whose first Chern class satisfies

$$c_1 = a\omega, \quad a < 0, \tag{4.1}$$

then c_1 is negative. By a theorem of Yau [10], we know that M admits an Einstein-Kaehler metric $\tilde{g}$. Since $\tilde{g}$ is Einstein-Kaehlerian and c_1 is negative, we have

$$c_1 = b\tilde{\omega} \tag{4.2}$$

for some $b < 0$, where $\tilde{\omega}$ is the fundamental class of $\tilde{g}$. By Theorem 2 of [4], we have

$$\tilde{\omega}^{n-2}c_2[M] \geqslant \frac{n}{2(n+1)}\tilde{\omega}^{n-2}c_1^2[M]. \tag{4.3}$$

Equalities (4.1) and (4.2) give $\tilde{\omega} = (a/b)\omega$. Thus, (4.3) implies

$$\omega^{n-2}c_2[M] \geqslant \frac{n}{2(n+1)}\,\omega^{n-2}c_1^2[M]. \tag{4.4}$$

On the other hand, from (3.7) and (3.8) we have

$$\omega^{n-2}c_1^2[M] = \frac{(n-2)!}{16\pi^2}\int(\varrho^2 - 2\|S\|^2)\,dV, \tag{4.5}$$

$$\omega^{n-2}c_2[M] = \frac{(n-2)!}{32\pi^2}\int(\varrho^2 - 4\|S\|^2 + \|R\|^2)\,dV. \tag{4.6}$$

Since g is a BK-metric, (4.6) implies

$$\omega^{n-2}c_2[M] = \frac{(n-2)!}{32\pi^2}\int\left(\frac{n(n+3)}{(n+1)(n+2)}\,\varrho^2 - \frac{4n}{n+2}\,\|S\|^2\right)dV.$$

Combining this with (4.5) we find

$$\omega^{n-2}c_2[M] - \frac{n}{2(n+1)}\,\omega^{n-2}c_1^2[M] = \frac{(n-2)!\,n}{32(n+1)(n+2)\pi^2}\int(\varrho^2 - 2n\|S\|^2)\,dV. \tag{4.7}$$

Since $\varrho^2 \leqslant 2n\|S\|^2$ and equality holds if and only if g is Einsteinian [4], (4.4) and (4.7) imply that g itself is an Einstein-Kaehler metric. Consequently, g has constant negative holomorphic sectional curvature. Therefore, M is covered biholomorphically by the unit ball D^n in C^n.

Conversely, if M is covered biholomorphically either by CP^n or D^n, then the standard Kaehler metrics on CP^n and D^n give the desired BK-metrics. This completes the proof of Theorems 2 and 3.

REFERENCES

[1] M. Berger, *Sur les variétés d'Einstein compactes*, C. R. IIIe Réunion Math. Expression Latine, Namur (1965), p. 35-55.

[2] B.-Y. Chen, *Some topological obstructions to Bochner-Kaehler metric and their applications*, Journal of Differential Geometry 13 (1978), p. 547–558.

[3] — *Chern classes and Kaehler metrics*, Bulletin of the Institute of Mathematics, Academia Sinica, 6 (1978), p. 259-270.

[4] — and K. Ogiue, *Some characterizations of complex space forms in terms of Chern classes*, The Quarterly Journal of Mathematics (Oxford) (3) 26 (1975), p. 459-464.

[5] S. Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor*, Annals of Mathematics 74 (1961), p. 570-574.

[6] K. Kodaira, *On Kähler varieties of restricted type*, ibidem 60 (1954), p. 28-48.

[7] M. Matsumoto and S. Tanno, *Kählerian spaces with parallel or vanishing Bochner curvature tensor*, Tensor, New Series, 27 (1973), p. 291-294.

[8] S. Webster, *On the pseudo-conformal geometry of a Kähler manifold*, Mathematische Zeitschrift 157 (1977), p. 265-270.

[9] K. Yano and S. Ishihara, *Kaehlerian manifolds with constant scalar curvature whose Bochner curvature tensor vanishes*, Hokkaido Mathematical Journal 3 (1974), p. 297-304.

[10] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proceedings of the National Academy of Sciences of the U.S.A. 74 (1977), p. 1798-1799.

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