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ON FRAMED LINKS OF PEIFFER IDENTITIES

BY

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0. Introduction. In this note we investigate relations between the framed links appearing in the description of some homotopies and sequences of Peiffer transformations acting on identities. Corollary to our main result (theorem in §3) gives an answer to the first of the two questions set by P. Stefan in [7]. This article has not been published yet so let us recall the corresponding fragment:

"To prove (or to disprove) the Whitehead conjecture it is sufficient to consider the case when the subcomplex L of a 2-dimensional CW complex K differs from K by a single 2-cell. Paint this extra 2-cell red and all the remaining 2-cells blue. Assume that $\pi_2 K = 0$ and let $p = (p_1, p_2, \ldots, p_n)$ be an identity amongst the blue relators (each p_i is a conjugate of a relator in L, or its inverse, and $p_1 p_2 \ldots p_n = 1$ in $\pi_1(K^1)$. Then p is equivalent to $\emptyset$ by Peiffer transformations in K. This gives a linkage consisting of a blue part X anchored to the 'floor' and a red part, a link Y, floating above. To prove that $\pi_2 L = 0$ we must show that p is equivalent to $\emptyset$ in L, that is that we can get rid of the red stuff.

If X and Y are geometrically unlinked, we are done. Otherwise, there seem to be two possible cases:

(1) X and Y are algebraically linked. Part of the problem is to make this idea precise – perhaps in terms of various degrees shades of blue, shades of red.

(2) X and Y are algebraically unlinked, but are still geometrically linked.".

We obtain that for any link appearing in the framework of the Whitehead conjecture the linking numbers between any connected toroidal component of the link and any part of the link with the same shadow is zero. Therefore one can assume that each connected component of this "red stuff" is algebraically unlinked with "blue stuff", so the first case does not occur.

In Section 1 we recall basic definitions and fix notations. In Section 2 we show how links appear in our situation and we make the description of framed links more precise for the purpose of the proof of our results which appear in Section 3.

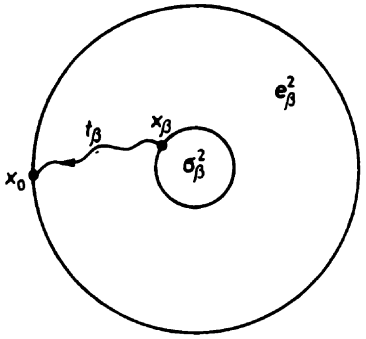
The author would like to thank Professor Joel Cohen for the invitation to the University of Maryland where this work was done.

1. Notations and definitions. Let K be a 2-dimensional connected CW complex with only one 0-cell x_0. (By contraction of the maximal tree contained in the 1-skeleton every 2-CW complex is homotopy equivalent to some K.) K has a presentation:

$$K = \bigvee_{x_\alpha} S^1 \cup \bigcup_{r_\beta} e^2_\beta$$

and so $\pi_1(K) = \langle x_\alpha | r_\beta \rangle$, $\alpha \in I$, $\beta \in J$ where I, J are some indexing sets. x_α corresponds to 1-cells of K and r_β are elements in a free group F freely generated by $\{x_\alpha | \ \alpha \in I\}$. Each r_β represents the way a given 2-cell e^2_β is attached to 1-skeleton of K. All x_α, $\alpha \in I$, are different, but there is no such assumption about r_β, $\beta \in J$.

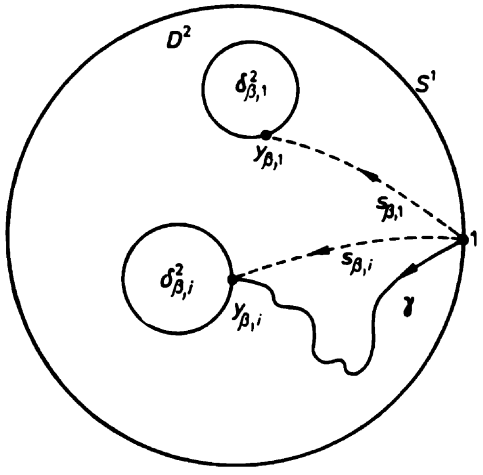
For each 2-cell e^2_β let σ^2_β be a 2-disc contained in the interior of e^2_β. Choose a point x_β on σ^2_β and a path t_β in $e^2_\beta \setminus \operatorname{int} \sigma^2_\beta$ going from x_β to x_0.



Let us denote by K_1 the closure of $K \setminus (\bigcup_{\beta \in J} \sigma^2_\beta)$. Let

$$f\colon (D^2, S^1, 1) \to (K, K_1, x_0)$$

be a representative of some element in $\pi_2(K, K_1, x_0)$. We can assume (using transversal arguments) that f is a transversal map on all 2-discs σ^2_β, i.e., each $f^{-1}(\sigma^2_\beta)$ is a finite set $\delta^2_{\beta,1}, \ldots, \delta^2_{\beta,s_\beta}$ of disjoint 2-discs in $D^2 \setminus S^1$, each $\delta^2_{\beta,i}$ mapped homeomorphically by f onto σ^2_β [3]. Let $y_{\beta,i} = f^{-1}(x_\beta) \in \delta^2_{\beta,i}$.



K_1 is homotopy equivalent to the 1-skeleton of K, so $\pi_1(K_1, x_0) = F = gp\{x_\alpha | \ \alpha \in I\}$. Each path γ in

$$D^2 \setminus [\bigcup_{\beta,i} (\delta_{\beta,i} \setminus \{y_{\beta,i}\}) \cup (S^1 \setminus \{1\})] = D_1$$

going from 1 to some $y_{\beta,i}$ is mapped by f onto a path in K_1 from x_0 to x_β. Therefore, the composition $f(\gamma) * t_\beta$ is a loop in K_1 which gives us some element $f^*(\gamma) \in F$.

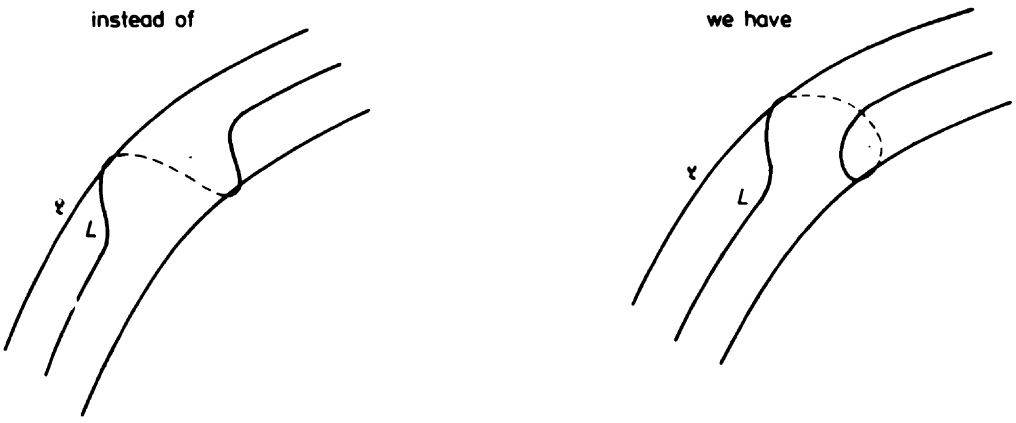
Let us choose a set of paths $s_{\beta,i} \subset D_1$ going from 1 to $y_{\beta,i}$ and meeting each other only at the point 1. Now $D_1 \setminus \bigcup_{\beta,i} s_{\beta,i}$ is an open 2-cell mapped into K_1, so the image by f of its boundary is contractible in K_1.

Now let us order the $s_{\beta,i}$ around 1 and relabel them as $s_1, \ldots, s_q$; we relabel the corresponding $\delta^2_{\beta,i}$ as $\delta^2_1, \ldots, \delta^2_q$ (also $y_{\beta,i}$ as $y_1, \ldots, y_q$) and we will write $\delta^2_{j,\beta}$ to denote that $f(\delta^2_{j,\beta}) = \sigma^2_\beta$.

2. Framed links with labels. Let f' be another transversal map f': $(D^2, S^1, 1) \to (K, K_1, x_0)$ such that $[f'] = [f]$ in $\pi_2(K, K_1, x_0)$; then there exists a homotopy H'

$$H'\colon (D^2 \times I, S^1 \times I, 1 \times I) \to (K, K_1, x_0).$$

H' can be deformed modulo $D^2 \times \{0, 1\}$ to H which is transversal with respect to $\{\sigma^2_\beta | \ \beta \in J\}$, i.e., for each $\beta \in J$, $H^{-1}(\sigma^2_\beta)$ is a disjoint union of solid tubes with ends attached to $D^2 \times \{0, 1\}$ and solid tori all labeled with β. Visually we can color these tubes and tori by β-color. The union $\bigcup_\beta H^{-1}(\sigma_\beta)$ forms a solid linkage $\mathfrak{L}$ in $D^2 \times I \setminus S^1 \times I$. $H^{-1}(x_\beta)$ is a union of paths lying on the boundary of $H^{-1}(\sigma^2_\beta)$ which gives us a certain framed link L_β colored by β-color. By the usual knot theory arguments we can choose a plane π orthogonal to $D^2 \times \{0\}$ such that projection of our link $L = \bigcup_\beta L_\beta$ from a point $1 \times \{0\}$ onto π is in regular position. By a slight deformation we can arrange our link so that each twist of the framed link has the following projection:



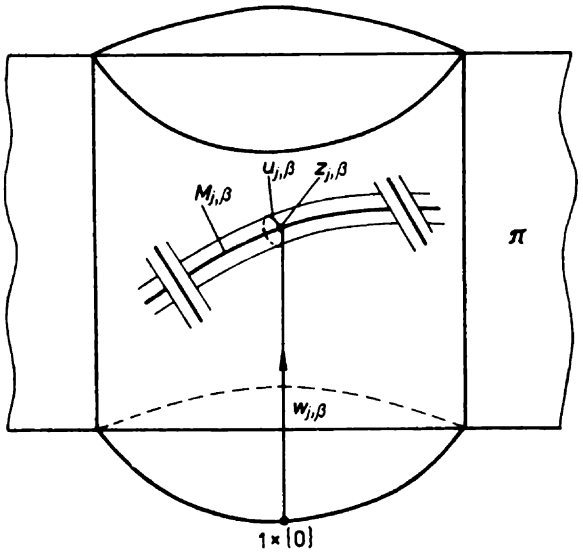
so the projection reflects the exact number of twists and, except when the link twists, it remains on the $1 \times \{0\}$-side of the boundary of the solid linkage Ω. We say that link is in *strong regular position* if its projection is as above.

For any path v from $1 \times \{0\}$ to some point of L lying in

$$D^2 \times I \setminus (\Omega \cup S^1 \times I) \stackrel{\text{def}}{=} A,$$

its image by H is in K_1, and when composed with the corresponding t_β gives us a loop in K_1, so a certain element $H^\#(v) \in F$. This element depends only on the homotopy class of the path v with fixed ends.

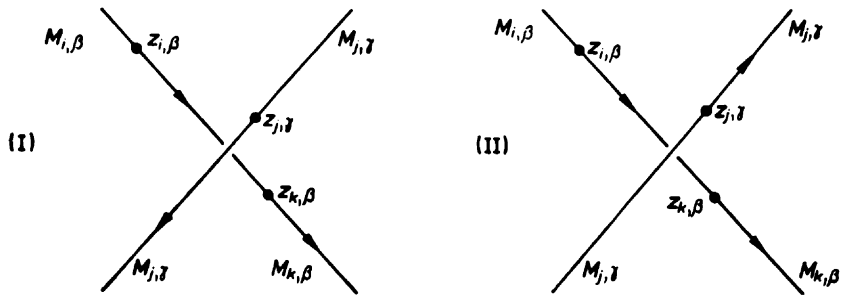
Let $\{M_{j,\beta}\}$ be the set of all overpasses of L. We give for each $M_{j,\beta}$ a label which belongs to $F \times \{r_\beta \mid \beta \in J\}$. Let $w_{j,\beta}$ be a direct path in A from $1 \times \{0\}$ to some point $z_{j,\beta} \in M_{j,\beta}$. We define a label of $M_{j,\beta}$ as $(H^\#(w_{j,\beta}), r_\beta)$. Let $u_{j,\beta}$ be a loop from $z_{j,\beta}$ to $z_{j,\beta}$ which goes once around the solid part of the linkage in the direction induced by the orientation of $\dot{e}^2_\beta$.



The image of the loop $w_{j,\beta} * u_{j,\beta} * w_{j,\beta}^{-1}$ in $\pi_1(K_1) = F$ is

$$H^\#(w_{j,\beta}) \cdot r_\beta \cdot H^\#(w_{j,\beta})^{-1}.$$

Any fixed orientation of $D^2 \times I$ and orientation of $u_{j,\beta}$ gives us an orientation of the framed link L. We have two types of oriented crossings:

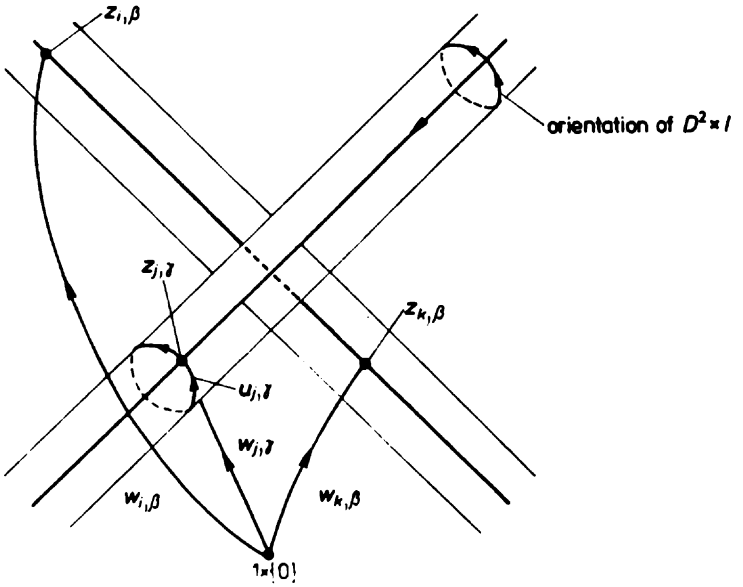


Let us investigate the relations between the labels of $M_{j,\beta}$ in each case. Let $D^2 \times I$ have the "right hand rule" orientation. Let $(z_{k,\beta}, z_{i,\beta})$ denote a path along the link.

We have

$$H^*(w_{k,\beta}) = H^*(w_{k,\beta} * (z_{k,\beta}, z_{i,\beta}))$$

because $H((z_{k,\beta}, z_{i,\beta})) = x_\beta$. The path $w_{k,\beta} * (z_{k,\beta}, z_{i,\beta})$ is homotopic to the path $w_{j,\gamma} * u_{j,\gamma}^{-1} * w_{j,\gamma}^{-1} * w_{i,\beta}$



so for (I)

$$H^*(w_{k,\beta}) = H^*(w_{j,\gamma}) \cdot r_\beta^{-1} \cdot H^*(w_{j,\gamma})^{-1} \cdot H^*(w_{i,\beta})$$

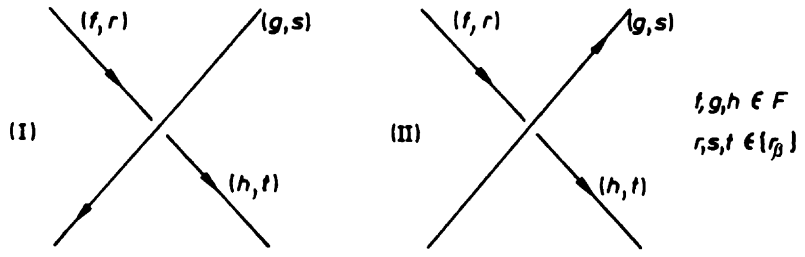
and similarly for (II)

$$H^*(w_{k,\beta}) = H^*(w_{j,\gamma}) \cdot r_\beta \cdot H^*(w_{j,\gamma})^{-1} \cdot H^*(w_{i,\beta})$$

in F.

We see that for any connected component of the framed link the H^*-part of its labels taken modulo the relations $\{r_\beta\}$ in the group $\pi_1(K) = \langle x_\alpha \mid r_\beta \rangle$ are the same. Let us take $|\pi_1(K)|$ different shadows in each color β. Any connected component can be given (g, β) shadow where $g \in \pi_1(K)$.

PROPOSITION. *Given a framed oriented link L in strong regular position in $D^2 \times I \setminus S^1 \times I$ with ends only on $D^2 \times (\{0\}, \{1\})$, together with given labeling of its overpasses by (f, r_β), $f \in F$, $\beta \in J$. Assume that for each oriented crossing:*

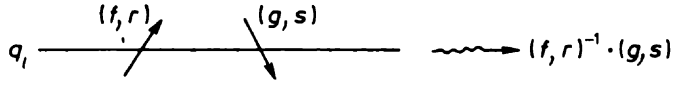


the relations (I) $(h, t) = (gs^{-1}g^{-1}f, r)$, (II) $(h, t) = (gsg^{-1}f, r)$ *hold. Then* L *can be realized as a homotopy between two functions* $f_0, f_1: (D^2, S^1, 1) \to (K, K_1, x_0)$ *determined by* $L \cap D^2 \times \{0\}$, $L \cap D^2 \times \{1\}$ *respectively.*

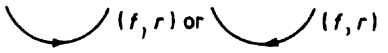
Proof. It is standard construction ([8], Lemma 3, pp. 251–252). ∎

Each link as above gives us a sequence of Peiffer transformations between two elements of the free semigroup $\mathfrak{F}(x_\alpha: r_\beta)$ freely generated by the set $F \times \{r_\beta\} \cup (F \times \{r_\beta\})^{-1}$ in the following way: By isotoping our link slightly we can arrange it in $D^2 \times I$ so that there exist levels $\{0 = q_0 < q_1 < \ldots < q_n = 1\}$ between which the link L involves just a single cap $\cap$, an overcrossing $\times$ or $\times$ or a cup $\cup$. Then work upward. At each level q_l let us read all labels (f, r_β) of overcrossings from left to right. Next if the orientation of the overpass crosses the level q_l downward we do not change (f, r_β), but if the orientation of this overpass goes upward let us take $(f, r_\beta)^{-1}$ instead.

Example:

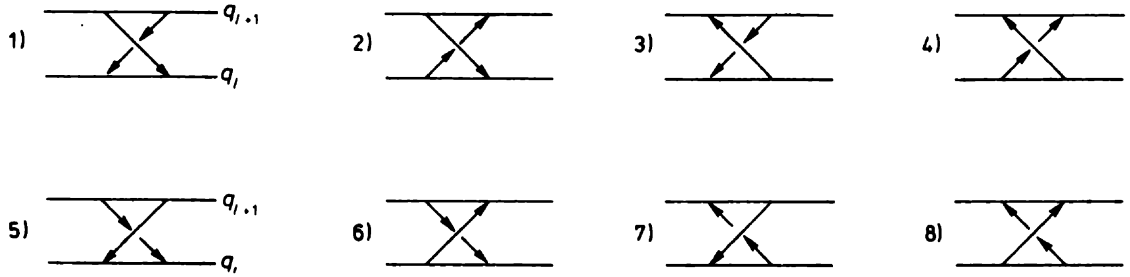


Each cup gives us a Peiffer insertion [2] of a subword $(f, r)(f, r)^{-1}$ or $(f, r)^{-1}(f, r)$ depending on the orientation



Similarly each cap gives us Peiffer deletion [2].

Now we have eight types of oriented crossings:



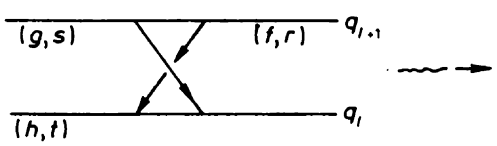
Using the fact that the labels satisfy relations (I) or (II) it is easy to see that each of the eight types of crossings gives us some Peiffer exchange

$$(f, r)^\varepsilon (g, s)^\delta \rightsquigarrow (g, s)^\delta (gs^{-\delta}g^{-1}f, r)^\varepsilon$$

or

$$(f, r)^\varepsilon (g, s)^\delta \rightsquigarrow (fr^\varepsilon f^{-1}g, s)^\delta (f, r)^\varepsilon, \qquad \varepsilon, \delta = \pm 1.$$

Example:



Because $h = gsg^{-1}f$, $t = r$ (II), we have $f = gs^{-1}g^{-1}h$, so

$$(h, t)(g, s) \rightsquigarrow (g, s)(gs^{-1}g^{-1}h, t).$$

Therefore we obtain Sieradski rules of labeling on abstract link ([7], §6). It is easy to see that each sequence of Peiffer transformations of elements from $\mathfrak{F}(x_\alpha\colon r_\beta)$ gives us an oriented framed link with labels as described in the proposition.

3. Toroidal components of link.

THEOREM. *Each connected toroidal component of a framed, oriented, labeled link L in strong regular position in $D^2 \times I$ with ends on $D^2 \times (\{0\}, \{1\})$ gives us an identity between relations $\{r_\beta |\ \beta \in J\}$.*

Proof. Let us choose a point p lying on some overpass $M_{j,\beta}$ of a given connected toroidal component. This overpass has the label $(H^\#(w_{j,\beta}), r_\beta)$ (proposition). Now let us move the point p along the component of the link according to the orientation of this component. Each crossing changes only the first part of the label $(H^\#(w_{j,\beta}), r_\beta)$ by multiplying it by some conjugates of the relations with respect to rules determined by (I) or (II). After a finite number of crossings (using the compactness of $D^2 \times I$), we return to the point p, but now the resulting label differs from $(H^\#(w_{j,\beta}), r_\beta)$ by the number of the twist in the framed component, so we can compare the resulting label with the label $(H^\#(w_{j,\beta}) \cdot r_\beta^k, r_\beta)$ where $k \in \mathbf{Z}$ denotes the number of twists (number of twist in accordance with the orientation − number of twist against given orientation of the linkage). This comparison gives us

$$H^\#(w_{j,\beta}) \cdot r_\beta^k = \prod_{i=1}^{l} f_i \cdot r_{\beta_i}^{\varepsilon_i} \cdot f_i^{-1} \cdot H^\#(w_{j,\beta})$$

or

$$\prod f_i \cdot r_{\beta_i}^{\varepsilon_i} \cdot f_i^{-1} \cdot H^\#(w_{j,\beta}) \cdot r_\beta^k \cdot H^\#(w_{j,\beta})^{-1} = 1$$

where

$$f_i \in F, \quad r_\beta \in \{r_\beta\}, \quad \varepsilon_i = \pm 1. \ \blacksquare$$

Assume now that K is *aspherical* and a map

$$f\colon (D^2, S^1, 1) \to (K, K_1, x_0)$$

is such that $f|_{S^1}$ is contractible in K_1 (so f represents an element in $\pi_2(K, x_0)$). Let H be any transversal nullhomotopy of f and L its oriented framed link with labels in the strong regular position.

Let us define the *linking number* of a connected toroidal component of the link L having shadow (g, β) with a part of the link L colored by shadow (h, γ). Choose a point p belonging to the connected component. Going along this component according to the orientation let us count the crossings:

The sum of given signs gives us the linking number.

Remark. The above definition is standard if the shadows (g, β) and (h, γ) are different. In the case where they are the same this linking number takes into account also the number of twists and the selfcrossings of our connected component.

COROLLARY. *Let K be aspherical CW-complex. The linking number of a connected toroidal component of the link L having shadow (g, β) with a part of the link colored by shadow (h, γ) is zero.*

Proof. K is aspherical, so any identity is a Peiffer identity. Therefore by the algebraic characterization of Peiffer identities for

$$\prod_{i=1}^{l} f_i \cdot r_{\beta_i}^{\varepsilon_i} \cdot f_i^{-1} \cdot \left(H^{\#}(w_{j,\beta}) \cdot r_\beta \cdot H^{\#}(w_{j,\beta})^{-1}\right)^k = \prod_{i=1}^{l+|k|} f_i \cdot r_{\beta_i}^{\varepsilon_i} \cdot f_i^{-1} = 1$$

where $r_{\beta_i} = r_\beta$, $f_i = H^{\#}(w_{j,\beta})$, $\varepsilon_i = \operatorname{sgn} k$ for $i = l+1, \ldots, l+|k|$ there is a pairing (i, j) of indices $i, j = 1, \ldots, l+|k|$ such that $r_i = r_j$, $\varepsilon_i = -\varepsilon_j$ and $f_i \equiv f_j \bmod \pi_1(k)$. Now going along connected component from the point p this pairing tells us that we meet each shadow twice with the different orientations of the crossings $(\varepsilon_i = -\varepsilon_j)$, so in the sum which defines the linking number the coefficients of these two crossings cancel and so all above linking numbers are zero. ∎

Remark. P. Stefan asked if a double Whitehead link can appear in a Whitehead conjecture setting. Considering this link, it is sufficient to consider only 2-complexes with two 2-cells, but every subcomplex with one 2-cell is aspherical, so there has to exist another nullhomotopy which is split by discs ([6], §4) and with the same bottom labels as in Whitehead double link.

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