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THE CARDINAL EQUATION $2m = m$

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Since it is known that the axiom of choice is equivalent to the statement "for all infinite cardinals m, $m^2 = m$", it is natural to ask the strength of the statement "for all infinite cardinals m, $2m = m$" (which we abbreviate to "$2m = m$"). In fact, very little is known, and the question as to whether $2m = m$ is equivalent to the axiom of choice is still open. In this paper * we show that none of Lévy's generalized axioms of dependent choice implies $2m = m$.

1. PRELIMINARIES

We work throughout in Zermelo-Fraenkel (ZF) set theory, which we take to include the axiom of foundation but not the axiom of choice.

Notations. ${}^{A}B$ is the set of functions with domain A and range a subset of B. $|X|$ is the cardinal of X; $A^B = |{}^{B}A|$. ON is the class of all ordinals. We take an *aleph* to be an initial ordinal. $S(X)$ is the power set of X; for $\varkappa$ an aleph $S_\varkappa(X) = \{Y \subseteq X\colon |Y| < \varkappa\}$. We set

$$\mathrm{Seq}_\varkappa(X) = \{f\colon f \in {}^{a}X \text{ for some } a < \varkappa\},$$
$$H_\varkappa(A, B) = \{f\colon \mathrm{dom}(f) \in S_\varkappa(A), \mathrm{ran}(f) \subseteq B\}.$$

$f|A$ is f restricted to A, that is $\{\langle a, b\rangle \in f\colon a \in A\}$.

Axioms of dependent choice.

DC. If A is a set and R is a binary relation such that

$$(\forall x \in A)(\exists y \in A)(xRy),$$

then there is an ω-sequence $(a_n)_{n<\omega}$ of elements of A such that $(\forall n < \omega)(a_n R a_{n+1})$.

* The result of this paper is taken from the author's Ph. D. thesis (University of Bristol 1971, supervised by Dr. F. Rowbottom). The author held a Monash University Travelling Scholarship when the research for the thesis was carried out.

DC^{α} (for α an ordinal). If A is a set and R is a binary relation with

$$(\forall \beta < \alpha)(\forall f \in {}^{\beta}A)(\exists x \in A)(fRx),$$

then there is $g \in {}^{\alpha}A$ such that $(\forall \beta < \alpha)\big((g|\beta)Rg(\beta)\big)$.

DC was introduced in [1], DC^{α} in [2]. In [2] it is shown that each DC^{α} is a consequence of the axiom of choice, that $DC \leftrightarrow DC^{\omega}$ (so DC^{α} is a generalization of DC), that $\alpha \geqslant \beta \to (DC^{\alpha} \to DC^{\beta})$ and that $DC^{|\alpha|} \to DC^{\alpha}$. The last two results show that we need only consider $DC^{\varkappa}$, where $\varkappa$ is an aleph.

Relative constructibility. We review this notion briefly. We write L for Gödel's constructible universe. If X is a transitive set, $L(X)$ is the smallest transitive class N such that $ON \subseteq N$, $X \in N$ and $\langle N, \in\rangle \models ZF$. There is a canonical function

$$F_0\colon\ L \times \mathrm{Seq}_{\omega}(X) \to L(X)$$

which is onto and is defined from X alone. Further, the definition of F_0 is absolute for any transitive proper class which contains X and satisfies ZF.

If X is not transitive, by $L(X)$ we mean $L\big(TC(X)\big)$, where

$$TC(X) = \{X\} \cup X \cup \left(\bigcup X\right) \cup \left(\bigcup\bigcup X\right) \cup \ldots$$

$TC(X)$ (the *transitive closure* of X) is the smallest transitive set with X as a member.

2. *CONSTRUCTION OF THE MODEL*

We will prove our theorem by constructing a model of set theory by means of Cohen's method of forcing. We follow Shoenfield's exposition [3].

Let M be a countable transitive model of $ZF + V = L$ and let $\varkappa$ be a (regular aleph)^{M}. We take as our notion of forcing

$$C = \big(H_{\varkappa}(\varkappa \times \varkappa, 2)\big)^{M}.$$

We define a partial order on C by setting $p \leqslant q \leftrightarrow p \supseteq q$. Let G be C-generic over M; $M[G]$ is a model of $ZF + AC$. In $M[G]$ set

$$G_i = \bigcup \{p(i)\colon\ p \in G\} \qquad \text{for } i < \varkappa,$$

where by $p(i)$ we mean $\{\langle j, k\rangle\colon\ \langle\langle i, j\rangle, k\rangle \in p\}$. Set $G^{*} = \{G_i\colon\ i < \varkappa\}$ and $H = \mathrm{Seq}_{\varkappa}(G^{*})$.

By standard methods we have

LEMMA 1. *In* $M[G]$

(i) *each* G_i *is a member of* ${}^{\varkappa}2$,

(ii) *the sequence* $(G_i)_{i<\varkappa}$ *is* 1-1,

(iii) *for each* $h \in (H_\varkappa(\varkappa, 2))^M$ *there is* $i < \varkappa$ *such that* $G_i \supseteq h$.

LEMMA 2. *Let* φ *be a ZF-formula,* $x \in M$. *Then* $M[G] \models$ *if* $s \in H$ *and* $G_i \notin \operatorname{ran}(s)$ *and* $\varphi(H, s, x, G_i)$, *then there is* $h \in (H_\varkappa(\varkappa, 2))^M$ *such that* $G_i \supseteq h$ *and*

$$\big(\forall r \in G^* - \operatorname{ran}(s)\big)\big(r \supseteq h \to \varphi(H, s, x, r)\big).$$

The model we are concerned with is $N = (L(H))^{M[G]}$. As noted in Section 1, we may construct in $M[G]$ (or in N) a canonical function

$$F_0\colon L \times \operatorname{Seq}_\omega(TC(H)) \to N$$

definable from $TC(H)$ (and thence from H). Now, if $x \in TC(H)$, $x = H$ or $x \in H$ or $x = G_i$ (for some $i < \varkappa$) or $x \in ON \times 2$ or $x \in ON$. So, by appropriate coding, we may replace F_0 by $F\colon L \times H \to N$. (For a finite sequence of ordinals and pairs of ordinals can be coded as an element of L, and a finite sequence of elements of H and elements of G^* can be coded as an element of H.)

If $z = F(x, s)$, we say z is *constructed by* H, x *and* s, or *constructible from* H *and* s.

We note that $(L)^{M[G]} = (L)^N = M$.

LEMMA 3. (i) M, N *and* $M[G]$ *have the same cofinality function and the same initial ordinals.*

(ii) *For any* $a < \varkappa$ *and* $x \in M$, $({}^a x)^M = ({}^a x)^N = ({}^a x)^{M[G]}$.

Proof. $H_\varkappa(\varkappa \times \varkappa, 2)$ can easily be shown to be $\varkappa$-closed and to satisfy the $\varkappa^+$-chain condition (see [3], Section 10, for these terms). The results for M and $M[G]$ then follow from theorems in [3], and the results for N are a consequence of the fact that $M \subseteq N \subseteq M[G]$.

3. MAIN RESULTS

LEMMA 4. *Let* A *be a set with at least two elements,* μ *a regular aleph. Suppose* $a < \mu$ *and let* $(s_i)_{i<a}$ *be a sequence of elements of* $\operatorname{Seq}_\mu(A)$. *Then there is a single element* s *of* $\operatorname{Seq}_\mu(A)$ *which codes the whole sequence* $(s_i)_{i<a}$.

LEMMA 5. *Suppose* $M[G] \models (X \subseteq N$ *and* $|X| < \varkappa)$. *Then* $x \in N$.

Proof. Assume that $M[G] \models X = \{t_i\colon i < \lambda\}$ for some $\lambda < \varkappa$. Since the axiom of choice holds in $M[G]$, we may in $M[G]$ choose sequences $(s_i)_{i<\lambda}$ and $(x_i)_{i<\lambda}$ such that $s_i \in H$, $x_i \in M$ and t_i is constructed by H, x_i and s_i. Now $(x_i)_{i<\lambda}$ is a single element of M (Lemma 3 (ii)), and $(s_i)_{i<\lambda}$ can be coded into a single sequence $s \in H$ (Lemma 4). So X is constructed by H, y, say, and s, where $y \in M$ incorporates $(x_i)_{i<\lambda}$ and "describes" the construction of X from s and $(x_i)_{i<\lambda}$. Thus $X \in N$.

THEOREM 1. *$N \models DC^{\alpha}$ for ordinals $\alpha < \varkappa$.*

Proof. Suppose R is a relation in N such that

$$N \models (\forall \beta < \alpha)(\forall f \in {}^{\beta}A)(\exists x \in A)(fRx).$$

Then

$$M[G] \models (\forall \beta < \alpha)(\forall f \in {}^{\beta}A)(\exists x \in A)(fRx),$$

for Lemma 5 shows that $({}^{\beta}A)^N = ({}^{\beta}A)^{M[G]}$. Since the axiom of choice, and thus DC^{α}, holds in $M[G]$, there is an $h \in {}^{\alpha}A$ such that $(h|\beta)Rh(\beta)$ for all $\beta < \alpha$. By Lemma 5, $h \in N$. So

$$N \models (\exists h \in {}^{\alpha}A)(\forall \beta < \alpha)\big((h|\beta)Rh(\beta)\big).$$

That is $N \models DC^{\alpha}$.

THEOREM 2. $N \models 2|G^*| > |G^*|$.

Proof. If $N \models 2|G^*| = |G^*|$, then in N there exist a set $X \subset G^*$ and a bijection $f\colon X \to G^*$ such that $N \models |G^* - X| = |G^*|$.

We suppose f is constructed by H, $x \in M$ and $s \in H$. Choose $G_j \in G^* - X$ and $G_i \in X$ such that $G_j \notin \operatorname{ran}(s)$ and $f(G_i) = G_j$. Note that the choice is always possible since

$$M[G] \models |\operatorname{ran}(s)| < \varkappa \quad \text{and} \quad |G^* - X| = \varkappa,$$

and that necessarily $G_i \neq G_j$.

Let φ be a formula such that

$$M[G] \models \varphi(H, s, x, G_i, G_j) \leftrightarrow N \models f(G_i) = G_j. \tag{1}$$

By Lemma 2, replacing s there by the sequence obtained by concatenating $\{G_i\}$ with s, there is an $h \in \big(H_{\varkappa}(\varkappa, 2)\big)^M$ such that $G_j \supseteq h$ and

$$M[G] \models \Big(\forall r \in G^* - \big(\operatorname{ran}(s) \cup \{G_i\}\big)\Big)\big(r \supseteq h \to \varphi(H, s, x, G_i, r)\big). \tag{2}$$

Now, we may easily construct in M a set A of elements of $\big(H_{\varkappa}(\varkappa, 2)\big)^M$ such that $|A| = \varkappa$, $h' \in A \to h' \supseteq h$ and $h', h'' \in A \to h'$ is incompatible with h'' (if $h' \neq h''$). Further, since M and $M[G]$ have the same alephs, $M[G] \models |A| = \varkappa$. So, by Lemma 2,

$$M[G] \models |\{r \in G^*\colon r \supseteq h\}| = \varkappa.$$

Since $M[G] \models |\operatorname{ran}(s)| < \varkappa$, we have

$$M[G] \models \Big|\big\{r \in G^*\colon r \supseteq h \text{ and } r \notin \big(\operatorname{ran}(s) \cup \{G_i\}\big)\big\}\Big| = \varkappa. \tag{3}$$

From (1), (2) and (3) we see that

$$N \models f \text{ takes infinitely many values at } G_i.$$

This is a contradiction, so $N \models 2|G^*| > |G^*|$.

Theorems 1 and 2 show that no particular axiom DC^a implies $2m = m$. However, we note that the axiom of choice is equivalent to $(\forall a)(DC^a)$ (see [2]), so certainly $(\forall a)(DC^a) \to 2m = m$.

Added in proof. It has recently been shown by G. Sageer (Ph. D. Thesis, The Hebrew University of Jerusalem, June 1973) that $2m = m$ is strictly weaker than the axiom of choice.

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