COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 2

ON WEAK AUTOMORPHISMS OF QUASI-LINEAR ALGEBRAS

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1. In this paper we shall examine weak automorphisms of quasi-linear algebras in which $+$ is an algebraic operation and 0 is the only algebraic constant. For the terminology and notation used here see [13], [3], [8].

An algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is said to be *quasi-linear* if the following conditions are satisfied:

(i) the set A is a subset of an Abelian group G,

(ii) for any operation $f \in \boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ there exist unary operations $f_1, f_2, \ldots, f_n$ on A (not necessarilly algebraic) such that

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j), \tag{1}$$

where the summation is the group operation in G,

(iii) there exists a one-to-one unary algebraic operation q such that the binary operation $r(x, y) = q(x) - q(y)$ is algebraic.

It follows from (iii) that zero-element of G is an algebraic constant in $\mathfrak{A}$.

As is known (cf. [3]), the class of quasi-linear algebras coincides with the introduced by E. Marczewski in [10] class of *separable variables algebras*. Among groups (or more generally, among n-groups; see [4]) only Abelian groups are separable variables algebras. Ω-group is a separable variables algebra iff it is an Abelian Ω-group (for definitions see [12], p. 115 and p. 147). Various variants of a notion of independence in separable variables algebras (in quasi-linear algebras) are examined in [2] and [6].

It is known (cf. [3], Theorem 3) that the algebra $(A; +)$ becomes an Abelian group under certain assumptions on the quasi-linear algebra $\mathfrak{A} = (A; \boldsymbol{F})$ (although $+$ need not to be an algebraic operation in $\mathfrak{A}$).

Now we shall prove

LEMMA 1. *If $\mathfrak{A} = (A; \boldsymbol{F})$ is a quasi-linear algebra such that*

(α) *the operation $+$ is algebraic in $\mathfrak{A}$,*

(β) *0 is the only algebraic constant in $\mathfrak{A}$,*
then every algebraic operation f can be expressed in the form

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n} h_j(x_j), \tag{2}$$

where h_j are endomorphisms of the semigroup $(A; +)$.

Proof. Any algebraic operation in a quasi-linear algebra has the form (1). Put $h_j(x) = f_j(x) - f_j(0)$. Then we have

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n} f_j(x_j) - \sum_{j=1}^{n} f_j(0) = \sum_{j=1}^{n} [f_j(x_j) - f_j(0)] = \sum_{j=1}^{n} h_j(x_j),$$

because, in virtue of (β), $f(0, \dots, 0) = 0$. Obviously, $h_j(0) = 0$ for every $j = 1, \dots, n$, and so all operations h_j are algebraic (although f_j need not to be algebraic operations). In view of the definition of a quasi-linear algebra, there exist unary operations g and h on A such that $f(x_1, x_2) \equiv h_j(x_1 + x_2) = h(x_1) + g(x_2)$, since $f \in A^{(2)}$. Therefore $h_j(x_1) = h(x_1) + g(0)$ and $h_j(x_2) = h(0) + g(x_2)$. Taking now into consideration $h(0) + g(0) = 0$ we infer that $h_j(x_1 + x_2) = h_j(x_1) + h_j(x_2)$. Hence h_j is an endomorphism of the semigroup $(A; +)$ and the proof of Lemma 1 is completed.

It seems worth to notice that if in a quasi-linear algebra with the only algebraic constant 0 is given an algebraic operation f of the form (1), and if some f_j is an endomorphism of the semigroup $(A; +, 0)$, then f_j is an algebraic operation in $\mathfrak{A}$. For that purpose it suffices to notice that $\sum_{i \neq j} f_i(0) = 0$.

2. Let f be an n-ary algebraic operation of an abstract algebra $\mathfrak{A}$, and let τ be a one-to-one transformation of A onto A. Let us define a function f^* by the equation

$$f^*(x_1, \dots, x_n) = \tau f(\tau^{-1} x_1, \dots, \tau^{-1} x_n). \tag{3}$$

If the mapping $f \to f^*$ is a one-to-one mapping of the class A onto itself, then the mapping τ is called a *weak automorphism.*

This notion (and, more generally, notions of a weak isomorphism and of a weak homomorphism) has been introduced by A. Goetz and E. Marczewski (see [8] and [11]). It turns out that Abelian groups have not essentially weak automorphisms. And even more, a group G, in which the square of every element belongs to its centre, does not have any weak automorphism which is not automorphism or anti-automorphisms [8]. Weak automorphisms of Boolean algebras and so-called Post algebras have been described in [14]. Description of weak automorphisms of integral domains appears in [7]. Recently, J. Dudek and E. Płonka have examined in [1] weak automorphisms of v^*-algebras (defined by E. Marczewski), a special case of which are linear spaces. The proofs of Theorems 2 and 3 of this paper take use of the same ideas as Dudek's proofs for linear spaces.

To start with we shall examine certain properties of the transformation * induced by a weak automorphism of quasi-linear algebras. Obviously, $0^* = 0$, and if $h \in A^{(1)}$ is an endomorphism of the group (semigroup) $(A;\, +)$, then also h^* is an endomorphism.

LEMMA 2. *If* $+$ *is an algebraic operation in the quasi-linear algebra* $\mathfrak{A}$ *with the one constant only, then*

$$(x+y)^* = x+y.$$

As a matter of fact, there exist unary operations g and h on A such that the equation $(x+y)^* = \tau(\tau^{-1}x+\tau^{-1}y) = g(x)+h(y)$ holds. Applying now Lemma 1, we infer that g and h are endomorphisms of the semigroup $(A;\, +)$. Putting $y = 0$ we get $x = g(x)$ for every $x \in A$. Similarly, for every $y \in A$, we obtain $y = h(y)$ and so the proof is completed.

LEMMA 3. *Equations*

$$r^*(x, y) = q^*(x)-q^*(y), \tag{4}$$

$$(-q)^* = -q^* \tag{5}$$

hold true in any quasi-linear algebra.

Indeed, let $r^*(x, y) = f(x)+g(y)$. Then

$$r^*(0, y) = \tau\big(-q(\tau^{-1}y)\big) = f(0)+g(y),$$
$$r^*(x, 0) = \tau\big(q(\tau^{-1}x)\big) = f(x)+g(0).$$

Now taking into account that $f(0)+g(0) = 0 = r(0, 0)$, we infer

$$r^*(x, y) = q^*(x)+(-q)^*(y). \tag{6}$$

Putting $x = y$ in (6) we get $r^*(x, x) = \tau r(\tau^{-1}x, \tau^{-1}x) = 0 = q^*(x)+$ $+(-q)^*(x)$. Hence (5) holds. From (5) and (6) follows (4).

LEMMA 4. *For every algebraic operation*

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j)$$

in a quasi-linear algebra with the only constant 0 *there is*

$$f^*(x_1, \ldots, x_n) = \Big(\sum_{j=1}^{n} f_j(x_j)\Big)^* = \sum_{j=1}^{n} \tau\big(f_j(\tau^{-1}x_j)-f_j(0)\big).$$

In fact, there exist operations g_j such that

$$f^*(x_1, \ldots, x_n) = \sum_{j=1}^{n} g_j(x_j).$$

Putting $x_1 = x_2 = \ldots = x_{j-1} = x_{j+1} = \ldots = x_n = 0$ and taking into account that $\sum_{j=1}^{n} f_j(0) = 0$ we obtain

$$\tau\big(f_j(\tau^{-1}x_j) - f_j(0)\big) = g_j(x_j) - g_j(0).$$

Since $\sum_{j=1}^{n} g_j(0) = 0$, our lemma easily follows.

In particular, if f_j are endomorphisms of the group (semigroup) $(A; +, 0)$, then it follows by Lemma 4 that

$$\Big(\sum_{j=1}^{n} f_j(x_j)\Big)^* = \sum_{j=1}^{n} f_j^*(x_j). \tag{7}$$

From Lemma 1 it follows that f_j^* also are endomorphisms of $(A; +)$.

3. If a quasi-linear algebra $\mathfrak{A}$ satisfies assumptions of Lemma 1 we shall denote by $E(\mathfrak{A})$ the set of endomorphisms of the semigroup $(A; +)$ which appear in the representation (2) of some algebraic operation.

Since A is a subset of an Abelian group with the group operation $+$, then, as is easy to notice by putting $h(-x) = -h(x)$ for $x \in A$, any endomorphism h of the semigroup $(A; +)$ can be extended to an endomorphism of the group $\mathfrak{G} = (A \cup (-A); +)$. And conversely, any endomorphism of the group $\big(A \cup (-A); +\big)$ is an endomorphism of the semigroup $(A; +)$.

By the remark following Lemma 1, with the respect to superpositions of functions, $E(\mathfrak{A})$ is a semigroup with 0 and 1, whence $E(\mathfrak{A}) \subset A^{(1)}$. And from the assumption that $+$ is an algebraic operation it follows that $E(\mathfrak{A})$ is a subsemiring [1] of the ring $E(\mathfrak{G})$ of all endomorphisms of the group $\mathfrak{G} = (A \cup (-A); +)$.

Moreover, as follows from the definition of a quasi-linear algebra, $E(\mathfrak{A})$ has the following additional properties:

$$\left.\begin{aligned} q(h_1) - q(h_2) &\in E(\mathfrak{A}) \\ -q(h_1) = q(-h_1) &\in E(\mathfrak{A}) \end{aligned}\right\} \quad \text{for all} \quad h_1, h_2 \in E(\mathfrak{A}).$$

THEOREM 1. *If a quasi-linear algebra $\mathfrak{A}$ satisfies conditions (α) and (β), then the set of all automorphisms of the semigroup $(A; +)$, which induce the one-to-one mappings of $E(\mathfrak{A})$ onto itself, coincides with the set of all weak automorphisms of $\mathfrak{A}$.*

Proof. If an algebra $\mathfrak{A}$ satisfies assumptions (α) and (β), then, by Lemma 1, any algebraic operation has the form (2). Let τ be an automorphism of the semigroup $(A; +)$ such that the induced mapping $*$ is

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[1] By a *semiring* we mean an algebra $(S; +, \cdot)$ such that $(S; +)$ and $(S; \cdot)$ are semigroups and the multiplication $\cdot$ is two-side distributive with the respect to the addition $+$. This notion was first introduced by H. S. Vandiver in [15] (see also [5]).

one-to-one and onto $E(\mathfrak{A})$. From (7) it follows then that the mapping $*$ is one-to-one and onto A.

Conversely, if τ is a weak automorphism of algebra $\mathfrak{A}$, then, by Lemma 2, τ is an automorphism of the semigroup $(A; +)$. By Lemma 1, unary algebraic operations are endomorphisms of $(A; +)$. τ induces one-to-one mapping of $E(\mathfrak{A})$ onto itself, because any weak automorphism maps the set of n-ary algebraic operations $A^{(n)}$ onto itself (for every $n = 0, 1, 2, \ldots$). Therefore the proof is completed.

One may conjecture that by the asumptions of Theorem 1 the mapping $*$ induced by a weak automorphism of the quasi-linear algebra $\mathfrak{A}$ is an automorphism of the semiring $E(\mathfrak{A})$. In fact, the more is true:

THEOREM 2. *If a quasi-linear algebra* $\mathfrak{A}$ *satisfies conditions* (α) *and* (β), *then the factor group* $\mathrm{Aut}^*\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ *can be isomorphically embedded into the group of automorphisms of the semiring* $E(\mathfrak{A})$ (*where* $\mathrm{Aut}^*\,\mathfrak{A}$ *and* $\mathrm{Aut}\,\mathfrak{A}$ *denote the groups of weak automorphisms and automorphisms of* $\mathfrak{A}$, *respectively*).

Proof. The idea of the proof is the same as that of the proof of a Dudek's theorem for linear spaces (see Corollary 1 of Theorem 1.2 and Theorem 1.1 in [1]). For every weak automorphism τ we define the mapping $\varphi_\tau\colon E(\mathfrak{A}) \to E(\mathfrak{A})$ by the equation $\varphi_\tau(h) = h^*$, $h \in E(\mathfrak{A})$. In other words, $\tau h(x) = \varphi_\tau(h)(\tau x)$. It is easy to verify that φ_τ is an automorphism of the semiring $E(\mathfrak{A})$ (we use Lemma 4 to verify that φ_τ is a homomorphism of the additive semigroup). Next we define mapping $\Phi\colon \mathrm{Aut}^*(\mathfrak{A}) \to \mathrm{Aut}\, E(\mathfrak{A})$ by the equation $\Phi(\tau) = \varphi_\tau$. It is easy to see that Φ is a homomorphism and that the kernel of this homomorphism, $\mathrm{Ker}\,\Phi$, coincides with $\mathrm{Aut}\,\mathfrak{A}$ (use equation (7) for the proof of inclusion $\mathrm{Aut}\,\mathfrak{A} \supset \mathrm{Ker}\,\Phi$), which completes the proof of Theorem 2.

It seems worth to notice that a linear space $\mathfrak{A} = (A; +, \lambda(\cdot);\ \lambda \in K)$ over a field K or, more generally, a unital left-module over a ring R with the unity 1 ($1x = x$ for every $x \in A$) is a quasi-linear algebra satisfying assumptions of Theorem 2. The elements of $E(\mathfrak{A})$ are in a one-to-one correspondence with the elements of the whole field K (or of the whole ring R, resp.). Thus

COROLLARY 1 (J. Dudek). *If* $\mathfrak{A} = (A; +, \lambda(\cdot); \lambda \in R)$ *is a unital left-module over* R, *then the factor group* $\mathrm{Aut}^*\,\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ *can be isomorphically embedded into the group of all automorphisms of* R.

Note that for a linear space $\mathfrak{A}$ over K the factor group $\mathrm{Aut}^*\,\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ is isomorphic to the group of all automorphisms of K. This fact is a consequence of the existence of a basis for every linear space and is a simple corollary of Theorem 1.2 in [1]. We say that the algebra $\mathfrak{A}$ has a *basis* if it has an independent set of generators (for the definitions see [13]). Similarly to Theorem 1.2 in [1] one can prove the following

THEOREM 3. *If a quasi-linear algebra $\mathfrak{A}$ has a basis and satisfies conditions* (α) *and* (β), *then the group* $\mathrm{Aut}^*\mathfrak{A}$ *is the normal product of the group* $\mathrm{Aut}\,\mathfrak{A}$ *and of the group of all automorphisms of the semiring* $E(\mathfrak{A})$.

Thus we have

COROLLARY 2. *If a quasi-linear algebra $\mathfrak{A}$ has a basis and satisfies* (α) *and* (β), *then the factor group* $\mathrm{Aut}^*\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ *is isomorphic to the group* $\mathrm{Aut}\,E(\mathfrak{A})$.

COROLLARY 3 (J. Dudek). *If $\mathfrak{A}$ is a unital left-module over R with a basis, then the factor group* $\mathrm{Aut}^*\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ *is isomorphic to the group* $\mathrm{Aut}\,R$.

As is known (see [9]), any group is isomorphic to the group of all automorphisms of a certain commutative ring with unity. Therefore, taking ring R with unity as a left-module over itself we have the following

COROLLARY 4. *For any group G there exists an algebra $\mathfrak{A}$ such that the factor group* $\mathrm{Aut}^*\mathfrak{A}/\mathrm{Aut}\,\mathfrak{A}$ *is isomorphic to the group G.*

Finally, we prove a theorem which is a generalization of the unpublished Dudek's theorem for linear spaces:

THEOREM 4. *If a quasi-linear algebra $\mathfrak{A}$ satisfies* (α), (β), *and*

(γ) *any algebraic operation in $\mathfrak{A}$ has the form* (2), *where h_j are, for every j, automorphisms of the semigroup* $(A;+)$,

then the group of all weak automorphisms of the algebra $\mathfrak{A}$ is the only maximal subgroup of the group of all automorphisms of the semi-group $\mathfrak{A}_0 = (A;+,0)$, *which contains as a normal subgroup the set $A(\mathfrak{A})$ of those automorphisms which appear in the form* (2) *of some algebraic operations.*

Proof. It is easy to notice that $A(\mathfrak{A})$ is a subgroup of $\mathrm{Aut}\,\mathfrak{A}_0$. Let us denote by $\mathscr{K}$ the family of those subgroups of $\mathrm{Aut}\,\mathfrak{A}_0$ which contain $A(\mathfrak{A})$ as a normal subgroup. Obviously, $A(\mathfrak{A})\in\mathscr{K}$. From Lemma 2 we infer that $\mathrm{Aut}^*\mathfrak{A}\subset\mathrm{Aut}\,\mathfrak{A}_0$. It is easy to check that $A(\mathfrak{A})$ is a normal subgroup of $\mathrm{Aut}^*\mathfrak{A}$, thus $\mathrm{Aut}^*\mathfrak{A}\in\mathscr{K}$. Now let G^* denote the group generated by the union of subgroups belonging to the family $\mathscr{K}$. Obviously, for any $G\in\mathscr{K}$ we have $G\subset G^*$. Therefore $\mathrm{Aut}^*\mathfrak{A}\subset G^*$. Now we shall prove that $G^*\subset\mathrm{Aut}^*\mathfrak{A}$. Let $g\in G^*$. Then there exist subgroups $G_i\in\mathscr{K}$ and elements $g_i\in G_i$ ($i=1,\ldots,n$) such that g is a superposition of g_i, i.e. $g=g_1\circ g_2\circ\ldots\circ g_n$. Hence g is a one-to-one mapping of A onto itself. It remains then to show that g induces a one-to-one mapping of A onto itself. For that purpose it suffices, by Theorem 1, to show that g induces a one-to-one mapping of $A(\mathfrak{A})$ onto itself, but this is an easy consequence of the fact that $A(\mathfrak{A})$ is a normal subgroup of G_i for $i=1,\ldots,n$. The theorem is thus proved.

Note that a linear space over a field K satisfies all assumptions of Theorem 3 (among algebras satisfying those assumptions are also, for instance, derived Abelian groups $\mathfrak{G} = (G; +, -, 0)$ and algebras $(G; \{+, -, 0\} \cup \mathrm{Aut}\,\mathfrak{G})$. Hence we have

COROLLARY 5 (J. Dudek). *If* $\mathfrak{A} = (A; +, -, 0, \lambda(-): \lambda \in K)$ *is a linear space over a field* K, *then the group* $Aut^* \mathfrak{A}$ *is the only maximal subgroup of the group of all automorphisms of the Abelian group* $\mathfrak{A}_0 = (A; +, -, 0)$, *which contains as a normal subgroup the set of all functions* $\lambda(x) = \lambda x$ *for every* $\lambda \neq 0$ $(\lambda \in K)$.

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Reçu par la Rédaction le 24. 5. 1968;
en version modifiée le 22. 4. 1970